Display
D

CANADIAN JOURNAL OF MATHEMATICS

Journal Canadien de Mathématiques

VOL. VII - NO. 2
1955

Two remarks on extreme forms — Martin Kneser 145
Note on extreme forms — E. S. Barnes 150
On integers n relatively prime to f(n) — Joachim Lambek and Leo Moser 155
On solutions of $x^d = 1$ in symmetric groups — Leo Moser and Max Wyman 159
The group ring of a class of infinite nilpotent groups — S. A. Jennings 169
Generalized matrix algebras — W. P. Brown 188
Some theorems on matrices with real quaternion elements — N. A. Wiegmann 191
A theorem concerning three fields — I. N. Herstein 202
On algebraic surfaces termwise invariant under cyclic collineations — J. C. Morelock and N. C. Perry 204
A theorem on fixed points of involutions in S^3 — Deane Montgomery and Hans Samelson 208
Full individual and class differentiation theorems in their relations to halo and vitali properties — C. A. Hayes, Jr. and C. Y. Pauc 221
Point-finite and locally finite coverings — Ernest Michael 275
On dynamical systems with one degree of freedom — C. R. Putnam 280
The propagation of a plane shock into a quiet atmosphere II — M. H. Martin 284

Published for

THE CANADIAN MATHEMATICAL CONGRESS

by the

University of Toronto Press

The chief languages of the *Journal* are English and French.

Manuscripts for publication in the *Journal* should be sent to the *Editor-in-Chief*, H. S. M. Coxeter, University of Toronto. Everything possible should be done to lighten the task of the reader; the notation and reference system should be carefully thought out. Every paper should contain an introduction summarizing the results as far as possible in such a way as to be understood by the non-expert.

All other correspondence should be addressed to the *Managing Editor*, G. de B. Robinson, University of Toronto.

The *Journal* is published quarterly. Subscriptions should be sent to the *Managing Editor*. The price per volume of four numbers is \$8.00. This is reduced to \$4.00 for individual members of recognized Mathematical Societies.

The Canadian Mathematical Congress gratefully acknowledges the assistance of the following towards the cost of publishing this Journal:

University of British Columbia

Carleton College
Ecole Polytechnique
Université Laval
Loyola College
University of Manitoba
McGill University
McMaster University
Université de Montréal
Queen's University
Royal Military College
St. Mary's University
University of Toronto

National Research Council of Canada
and the
American Mathematical Society

AUTHORIZED AS SECOND CLASS MAIL, POST OFFICE DEPARTMENT, OTTAWA

L

S

5

I

TWO REMARKS ON EXTREME FORMS

MARTIN KNESER

1. Introduction. The following remarks concern two different parts of the theory of extreme quadratic forms. In §2, I shall give a new proof for the theorem of Voronoï **(4)**, which asserts that a form is extreme if and only if it is both perfect and eutactic. (For the definitions see, e.g., Coxeter **(2)** or the text below.) There is indeed a comparatively simple proof in Bachmann's *Zahlentheorie*, IV, 2. For two reasons, however, it may not be useless to communicate another proof here. First, I shall prove the necessity and the sufficiency in one step, and second, as a consequence, my proof requires a minimum of calculation. In §3, I shall add a new senary extreme form to Hofreiter's list **(3)** and Coxeter's paper **(2)**. This form was found independently by Barnes (see the succeeding paper **(1)**). I therefore restrict myself to determining its group of automorphs, which is not done in Barnes's paper.

2. Voronoï's theorem. In this section, we denote: real symmetric $(n \times n)$-matrices by capital letters $A, B, \ldots$, the $\frac{1}{2}n(n+1)$-dimensional space of these matrices by R; vectors, considered as $(n \times 1)$-matrices, by small letters $x, y, \ldots$; the transposition of a matrix by a prime, and real numbers by Greek letters. Let $x'Ax$ be a positive definite quadratic form, and $\mu(A)$ the minimum of $x'Ax$ taken over all x with integer coordinates not all 0. This minimum is attained at a finite number of vectors, called minimal vectors; for these we shall reserve the letter m. The form $x'Ax$ is called *extreme* if the quotient

$$\delta(B) = \mu(B)\det(B)^{-1/n}$$

has a local maximum at $B = A$, i.e., if $\delta(A + C) < \delta(A)$ for all matrices $C \in R$ whose elements are sufficiently small. Since $\delta(\lambda B) = \delta(B)$, we may restrict C to a hyperplane through the origin not containing A, e.g. the hyperplane H of R defined by $\operatorname{tr}(A^{-1}C) = 0$. Continuity shows that, for small C, $A + C$ is positive definite and the minimal vectors of $A + C$ are contained among those of A. Hence $\delta(A + C) < \delta(A)$ means that for at least one minimal vector m of A, the inequality

$$m'(A + C)m.\det(A + C)^{-1/n} < m'Am.\det(A)^{-1/n} \tag{1}$$

holds; or, what is the same, the union of the regions K_m of C's such that $A + C$ is positive definite and (1) holds, contains a neighbourhood of the origin O in R; or, finally

(a) $\bigcup_m H \cap K_m$ contains a neighbourhood of O in H.

Now, the following lemma shows that K_m is convex and $H \cap K_m$ is strictly convex.

Lemma. *Let m be a fixed vector and $\lambda > 0$. The set K of positive definite symmetric matrices C satisfying the inequality $m'Cm < \lambda \det(C)^{1/n}$ is a convex cone with its vertex at the origin O. If H is a hyperplane not containing O, $H \cap K$ is strictly convex (i.e., if C and D are in $H \cap K$ and if $0 < \tau < 1$, then $\tau C + (1 - \tau)D$ is an inner point of $H \cap K$).*

Proof. We have to show that if C and D are in K, so is $\tau C + (1 - \tau)D$, and that the latter is an inner point unless $D = \kappa C$. Since K obviously is a cone, we may replace D by a scalar multiple and so may assume

$$m'Cm = m'Dm = m'(\tau C + (1 - \tau)D)m.$$

It remains to prove that the inequalities

$$\det(C)^{1/n} > \mu = m'Cm.\lambda^{-1}, \quad \det(D)^{1/n} > \mu$$

imply $\det(\tau C + (1 - \tau)D)^{1/n} > \mu$ with equality only if $D = C$. This is well known. It is proved by transforming C and D simultaneously to diagonal form and then applying the inequality

$$\prod_{i=1}^{n} (\tau\gamma_i + (1 - \tau)\delta_i)^{1/n} > \tau \prod_{i=1}^{n} \gamma_i^{1/n} + (1 - \tau) \prod_{i=1}^{n} \delta_i^{1/n}.$$

Moreover, K_m possesses a tangent hyperplane at the origin. This will be shown by developing (1) into a power-series in the elements of C and taking the linear terms only. In point of fact, write (1) in the form

$$1 + m'Cm\,\mu(A)^{-1} < \det(E + A^{-1}C)^{1/n}.$$

For the half-space determined by the tangent hyperplane and containing K_m, we then obtain the linear inequality

$$m'Cm\,\mu(A)^{-1} < \frac{1}{n}\operatorname{tr}(A^{-1}C).$$

Restricting C to H, we obtain

$$l_m(C) = m'Cm < 0.$$

Next we show that (a) is equivalent to

(b) The half-spaces $l_m(C) < 0$ cover the whole space H except O.

First suppose (a) holds. Let $C \neq O$ be any point of H. It follows from (a) that there exist an m and a $\lambda > 0$ such that λC is in $L_m = H \cap K_m$. Since L_m is strictly convex, μC is an inner point of L_m if $0 < \mu < \lambda$, and thus

$$l_m(C) = \mu^{-1} l_m(\mu C) < 0,$$

which proves (b). Conversely, suppose (b) holds. Let S be the unit sphere in H. For every $C \in S$ there is an m with $l_m(C) < 0$. Since $l_m = 0$ is the tangent hyperplane of L_m at O, there exists a $\lambda > 0$ such that λC is an inner point of

L_m. Then there is a neighbourhood U of C on S such that $\mu D \in L_m$ for every $D \in U$ and $0 < \mu < \lambda$. As S is compact, it is covered by a finite number of U's. Denote by λ_0 the minimum of the corresponding λ's. Then the solid sphere of radius λ_0 is contained in $\bigcup_m L_m$, which proves (a). Transforming (b) into a statement about the complementary half-spaces $l_m(C) > 0$, we get

(c) O is the only point of H common to all the half-spaces $l_m(C) > 0$.

Now, let H' be the space of linear forms on H, and $M \subset H'$ the convex set generated by O and the l_m. Since the elements l of any hyperplane in H' passing through the origin satisfy an equation $l(C) = 0$ with some fixed $C \neq O$, the statement (c) means that there is no hyperplane through O in H' leaving M on one side. Since M is convex, a neighbourhood of O in H' is contained in M. This assertion can be divided into two parts. First, H' is the linear space generated by M, i.e., M is not contained in any hyperplane in H'. This means that there is no hyperplane passing through O and the l_m. Second, O is an inner point of M relative to the linear space generated by M, i.e., there are numbers ρ_0, $\rho_m > 0$ with sum 1, such that $O = \rho_0.O + \sum_m \rho_m l_m$. We may multiply this equation by an arbitrary positive number, and so the condition that the sum of the coefficients is 1 may be dropped. We have the following two statements which, together, are equivalent to (c).

(d) The equations $l_m(C) = 0$ have no common solution $C \neq O$ in H.

(e) There exist positive numbers ρ_m such that $\sum_m \rho_m l_m(C) = 0$ for all C in H.

Since H is defined by $\operatorname{tr}(A^{-1}C) = 0$, (e) is equivalent to

$$\sum_m \rho_m\, m'Cm = \sigma \operatorname{tr}(A^{-1}C) \qquad \text{for all } C \text{ in } R, \tag{2}$$

with some constant σ. Putting $C = A$ we get $\sigma > 0$. If $C = x.x'$ we obtain

$$\sum_m \rho_m (m'x)^2 = \sigma \operatorname{tr}(A^{-1}xx') = \sigma \operatorname{tr}(x'A^{-1}x) = \sigma\, x'A^{-1}x. \tag{3}$$

This means that A is eutactic. Conversely (2) follows from (3), since every symmetric matrix C is a linear combination of matrices of the type xx' (this is nothing else than the well-known theorem that every quadratic form is a linear combination of squares of linear forms). Finally, (d) and (e) also imply that the equations $m'Cm = 0$ have no common solution in R, except $C = O$. For, if C is a solution, (2) implies $C \in H$ and then $C = O$ according to (d). This means that A is perfect and so we have proved

VORONOI'S THEOREM. *A positive definite quadratic form is extreme if and only if it is both perfect and eutactic.*

3. The new senary extreme form. The senary form

$$x'Ax = \sum_{i=1}^{3} (x_i^2 - x_i x_{i+3} + x_{i+3}^2) + \left(\sum_{i=1}^{6} x_i\right)^2$$

with the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & \frac{1}{2} & 1 & 1 \\ 1 & 2 & 1 & 1 & \frac{1}{2} & 1 \\ 1 & 1 & 2 & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & 2 & 1 & 1 \\ 1 & \frac{1}{2} & 1 & 1 & 2 & 1 \\ 1 & 1 & \frac{1}{2} & 1 & 1 & 2 \end{pmatrix}$$

has determinant $2^{-6}.3^2.13$ and minimum 2. There are 21 pairs $\pm m_i$ of minimal vectors. First, the pairs $\pm m_i$ $(i = 1, \ldots, 6)$ of unit vectors with ith coordinate ± 1, all others 0; second, 12 pairs $\pm m_i$ $(i = 7, \ldots, 18)$ with $x_k = 1, x_l = -1$ $(k \equiv l \bmod 3)$ and $x_j = 0$ if $j \neq k, l$; third, three pairs $\pm m_i$ $(i = 19, 20, 21)$ with coordinates obtained from $x_1 = x_4 = 1$, $x_2 = x_5 = -1$, $x_3 = x_6 = 0$ by permuting the values $1, -1, 0$. That A is perfect is seen by an easy calculation. That it is eutactic follows from the identity

$$(4) \qquad 39\, y'Ay = 6\sum_{i=1}^{6} (m_i'Ay)^2 + 5\sum_{i=7}^{18} (m_i'Ay)^2 + 7\sum_{i=19}^{21} (m_i'Ay)^2$$

which, by the substitution $y = A^{-1}x$, yields a formula of type (3). The representation (4) of $y'Ay$ as a linear combination of the squares of the linear forms $m_i'Ay$ is unique. This fact is important for the determination of the group of automorphs of A, i.e., of those unimodular matrices U which transform A into itself: $U'AU = A$. At the first glance there is a group G of 48 automorphs, namely those permutations of the variables x_i, which change each pair (x_i, x_{i+3}) into a pair of the same kind. We shall show that G combined with $\pm E$ is the whole group of automorphs of A. Obviously every U permutes the minimal vectors and preserves the scalar products $m_i'Am_k$. Now U transforms the representation (4) into another of the same kind, with Um_i instead of m_i. Since (4) is unique and the first coefficient, 6, is different from the two others, 5 and 7, U permutes the pairs $\pm m_i$ $(i = 1, \ldots, 6)$ amongst each other. This permutation is such that any quadruple $\pm m_i$, $\pm m_{i+3}$ changes into one of the same kind, because $m_i'Am_k = 1$ or $\neq 1$ according as $i \not\equiv k$ or $i \equiv k \bmod 3$. Hence we can write every automorph as a product of an element of G by an automorph V with $Vm_i = \pm m_i$ $(i = 1, \ldots, 6)$. As $m_i'Am_k \neq 0$, the sign must be the same for all i and so $V = \pm E$. Therefore the total group of automorphs is the direct product of G and the cyclic group $\pm E$ of order two.

As we have seen, the group of automorphs is not transitive on the minimal vectors, contrary to most other known extreme forms (2), nor is it irreducible. The irreducible subspaces are: One of dimension one, generated by

$$\sum_{i=1}^{6} m_i;$$

one of dimension two, generated by m_{19}, m_{20}, m_{21}; one of dimension three, generated by $n_1 = m_1 - m_4$, $n_2 = m_2 - m_5$, $n_3 = m_3 - m_6$. The representation

of G in the third subspace is isomorphic. G induces all transformations of the form $n_i \to \pm n_k$, $i \to k$ being any permutation and $\pm$ any combination of signs. So G is isomorphic to the extended octahedral group. This completes the determination of the structure of the group of automorphs.

References

1. E. S. Barnes, *Note on extreme forms*, Can. J. Math. *7* (1955), 150–154.
2. H. S. M. Coxeter, *Extreme forms*, Can. J. Math. *3* (1951), 391–441.
3. N. Hofreiter, *Ueber Extremformen*, Monatsh. Math. Phys. *40* (1933), 129–152.
4. G. Voronoï, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques*, J. reine angew. Math. *133* (1908), 97–178.

Universität Heidelberg

NOTE ON EXTREME FORMS

E. S. BARNES

Let $f(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$ be a positive definite quadratic form of determinant $D = |a_{ij}|$, and let M be the minimum of f for integral $x_1, \ldots, x_n$ not all zero. The form f is said to be *extreme* if the ratio M^n/D does not increase when the coefficients a_{ij} of f suffer any sufficiently small variation.

All extreme forms in n variables are known for $n \leq 5$. Hofreiter **(2)** investigated the problem of finding all extreme forms in 6 variables and listed four forms; but, as is pointed out by Coxeter **(1)**, one of these (F_4) is certainly not extreme. Coxeter **(1)** actually finds independently four extreme forms (including three of the four listed by Hofreiter) and makes the reasonable suggestion that the list is now complete.

The main purpose of this note is to show that there is an extreme form in 6 variables not given by these authors, namely

$$f(x_1, \ldots, x_6) = \left(\sum_{i=1}^{6} x_i\right)^2 + \sum_{j=1}^{3} \phi(x_j, x_{3+j}), \tag{1}$$

where generally

$$\phi(x, y) = x^2 - xy + y^2,$$

for which

$$M = 2,\ D = \frac{13 \cdot 3^3}{2^6}.$$

The form (1) is the particular case $n = 2r = 6$ of the form

$$f(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i\right)^2 + \sum_{j=1}^{r} \phi(x_j, x_{r+j}) + \sum_{k=2r+1}^{n} x_k^2, \tag{2}$$

where $n \geq 2r \geq 2$ (and the last sum is empty if $n = 2r$). I show here that *the form* (2) *is extreme if and only if*

$$4r - 2 \geq n \geq 2r \geq 6. \tag{3}$$

We first examine all integral sets $x_1, \ldots, x_n$, not all zero, for which

$$f \leq 2. \tag{4}$$

Noting that

$$\phi(x, y) \begin{cases} = 0 \text{ if } & (x, y) = (0, 0), \\ = 1 \text{ if } & \pm(x, y) = (1, 0),\ (0, 1) \text{ or } (1, 1), \\ \geq 3 \text{ otherwise}, & \end{cases} \tag{5}$$

Received October 1, 1954. The author would like to thank Professor Coxeter and Professor Mordell for their comments on an earlier draft of this note.

we see that (4) implies that

$$\phi(x_j, x_{r+j}) \leq 1 \qquad (j = 1, \ldots, r), \tag{6}$$

and that equality can hold in (6) for at most two values of j.

If, say, $\phi(x_1, x_{r+1}) = \phi(x_2, x_{r+2}) = 1, \phi(x_j, x_{r+j}) = 0$ $(j \geq 3)$, then

$$x_j = x_{r+j} = 0 \ (3 \leq j \leq r),$$

and (4) requires $2f = 2$, $x_k = 0$ $(k \geq 2r+1)$, $x_1 + x_2 + x_{r+1} + x_{r+2} = 0$. Using (5), we see that all possible sets $(x_1, x_2, x_{r+1}, x_{r+2})$ are $\pm(1, -1, 0, 0)$, $\pm(1, 0, 0, -1)$, $\pm(0, 1, -1, 0)$, $\pm(0, 0, 1, -1)$, $\pm(1, -1, 1, -1)$.

If only one $\phi(x_j, x_{r+j})$ is non-zero, say $\phi(x_1, x_{r+1})$, then $x_j = x_{r+j} = 0$ $(2 \leq j \leq r)$ and $x_{2r+1}^2 + \ldots + x_n^2 \leq 1$. There are thus two possibilities:

(i) $x_{2r+1} = \ldots = x_n = 0, \quad f = 2x_1^2 + 2x_{r+1}^2 + x_1 x_{r+1},$

and trivially $f \geq 2$, with equality only if

$$(x_1, x_{r+1}) = \pm(1, 0), \pm(0, 1);$$

(ii) $x_k = \pm 1$ for some $k \geq 2r + 1$,

$$f = (x_1 + x_{r+1} + x_k)^2 + 2,$$

and so $f \geq 2$, with equality only if

$$(x_1, x_{r+1}, x_k) = \pm(1, 0, -1), \pm(0, 1, -1).$$

If finally all $\phi(x_j, x_{r+j})$ are zero, then

$$f = \left(\sum_{k=2r+1}^{n} x_k\right)^2 + \sum_{k=2r+1}^{n} x_k^2,$$

and it is easily seen that $f \geq 2$, with equality only if some x_k is ± 1 and the rest zero, or if some pair $(x_k, x_l) = \pm(1, -1)$ and the remaining x_k are zero.

We have thus shown that $M = 2$. Also, if with each pair $\pm(x_1^{(t)}, \ldots x_n^{(t)})$ of minimal vectors (i.e., sets with $f = 2$), we associate the linear form

$$\lambda_t = \lambda_t(y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i^{(t)} y_i,$$

we see (using the obvious symmetries of f) that there are precisely

$$s = \tfrac{1}{2}n(n+1) + \tfrac{1}{2}r(r-3) \tag{7}$$

such linear forms, given by

$$\begin{array}{lr} y_i & (1 \leq i \leq n), \\ y_j - y_k & (1 \leq j < k \leq n, k \neq r+j;\ j = 1, \ldots, r), \\ y_l - y_m + y_{r+l} - y_{r+m} & (1 \leq l < m \leq r). \end{array} \tag{8}$$

If $r \geq 3$, it is easily verified that f is uniquely determined by the fact that it has minimum 2 and associated linear forms (8); thus f is *perfect* (in the sense of Voronoï **(3)**). If however $r < 3$, (7) gives $s < \tfrac{1}{2}n(n+1)$ and so f

cannot be perfect. We have therefore shown that f *is perfect if and only if* $r \geqslant 3$.

Now Voronoï **(3)** has shown that a form is extreme if and only if it is perfect and eutactic. We therefore consider next the problem of deciding when f is eutactic, that is to say, when its adjoint $F(y_1, \ldots, y_n)$ is expressible as

$$F(y_1, \ldots, y_n) = \sum_{t=1}^{s} \rho_t \lambda_t^2, \rho_t > 0 \qquad (t = 1, \ldots, s). \tag{9}$$

The labour of calculating F (and the determinant D of f) may be lightened by using the following method:

The form

$$g(z_1, \ldots, z_n) = (\alpha_1 z_1 + \ldots + \alpha_n z_n)^2 + \sum_{i=1}^{n} z_i^2 \tag{10}$$

is easily found to have determinant

$$D(g) = 1 + \sum_{i=1}^{n} \alpha_i^2, \tag{11}$$

and adjoint a multiple of

$$G(z_1, \ldots, z_n) = \sum c_{ij} z_i z_j \tag{12}$$

with

$$c_{ii} = 1 + \sum_{\substack{k=1 \\ k \neq i}}^{n} \alpha_k^2, \; c_{ij} = -\alpha_i \alpha_j \qquad (i \neq j).$$

Under the linear transformation T defined by

$$\left.\begin{aligned} x_j &= z_j + (1/\sqrt{3}) z_{r+j} \\ x_{r+j} &= (2/\sqrt{3}) z_{r+j} \end{aligned}\right\} \qquad (j = 1, \ldots, r),$$
$$x_k = z_k \qquad (k = 2r+1, \ldots, n),$$

f in (2) is reduced to the form (10) with

$$\alpha_i = 1 \;\; (1 \leqslant i \leqslant r, 2r+1 \leqslant i \leqslant n), \quad \alpha_i = \sqrt{3} \;\; (r+1 \leqslant i \leqslant 2r), \tag{13}$$

so that

$$1 + \sum_{i=1}^{n} \alpha_i^2 = n + 2r + 1. \tag{14}$$

Since T has determinant $(2/\sqrt{3})^r$, it follows from (11) and (14) that f has determinant

$$D = (\tfrac{3}{4})^r (n + 2r + 1). \tag{15}$$

Finally, a straightforward multiplication of matrices now shows that $F(y_1, \ldots, y_n)$ is a multiple of $\sum b_{ij} y_i y_j$ with

$$b_{ii} = \begin{cases} 4(n + 2r - 2), & 1 \leq i \leq 2r, \\ 3(n + 2r), & i > 2r, \end{cases}$$

$$(16) \qquad b_{ij} = \begin{cases} -12, & 1 \leq i < j \leq 2r, j - i \neq r, \\ 2(n + 2r - 5), & 1 \leq i < j \leq 2r, j - i = r, \\ -6, & i \leq 2r, j > 2r, \\ -3, & j > i > 2r. \end{cases}$$

Corresponding to the enumeration (8) of the associated linear forms, we write (9) as

$$(17) \qquad F(y_1, \ldots, y_n) = \sum \rho_i y_i^2 + \sum \sigma_{jk}(y_j - y_k)^2 + \sum \tau_{lm}(y_l - y_m + y_{r+l} - y_{r+m})^2,$$

where the suffixes have the ranges given in (8), and solve (17) for the $s = \frac{1}{2}n(n + 1) + \frac{1}{2}r(r - 3)$ coefficients $\rho_i, \sigma_{jk}, \tau_{lm}$.

First, we have immediately

$$(18) \qquad \sigma_{jk} = -b_{jk} = 3, \qquad k > j > 2r,$$

$$(19) \qquad \sigma_{jk} = -b_{jk} = 6, \qquad k > 2r \geq j.$$

The coefficient of $2y_jy_k$ for $1 \leq j < k \leq 2r$, $k - j \neq r$ is $-\sigma_{jk} - \tau_{lm}$, where $l = j$ or $j - r$, $m = k$ or $k - r$; hence we have

$$(20) \qquad \sigma_{jk} = \sigma_{j,r+k} = \sigma_{r+j,r+k} = 12 - \tau_{jk}, \qquad 1 \leq j < k \leq r.$$

The coefficient of $2y_j y_{r+j}$, for $1 \leq j \leq r$, is

$$(21) \qquad \tau_{1j} + \ldots + \tau_{j-1,j} + \tau_{j,j+1} + \ldots + \tau_{jr} = 2(n + 2r - 5) \qquad (j = 1, \ldots, r).$$

The coefficient of y_1^2 is

$$\begin{aligned} &\rho_1 + \sigma_{12} + \ldots + \sigma_{1n} + \tau_{12} + \ldots + \tau_{1r} \\ &\quad = \rho_1 + (12 - \tau_{12}) + \ldots + (12 - \tau_{1r}) + (12 - \tau_{12}) + \ldots + (12 - \tau_{1r}) \\ &\qquad\qquad + 6(n - 2r) + \tau_{12} + \ldots + \tau_{1r} \\ &\quad = \rho_1 + 12(2r - 2) + 6(n - 2r) - (\tau_{12} + \ldots + \tau_{1r}) \\ &\quad = \rho_1 + 4n + 8r - 14, \end{aligned}$$

using (19), (20) and (21); since $b_{11} = 4(n + 2r - 2)$, it follows that $\rho_1 = 6$. The same argument gives

$$(22) \qquad \rho_i = 6 \qquad 1 \leq i \leq 2r.$$

The coefficient of y_{2r+1}^2 is

$$\begin{aligned} &\rho_{2r+1} + \sigma_{1,2r+1} + \ldots + \sigma_{2r,2r+1} + \sigma_{2r+1,2r+2} + \ldots + \sigma_{2r+1,n} \\ &\qquad = \rho_{2r+1} + 6(2r) + 3(n - 2r - 1), \end{aligned}$$

using (18) and (19); since $b_{2r+1,2r+1} = 3(n + 2r)$, it follows that $\rho_{2r+1} = 3$. The same argument gives

$$(23) \qquad \rho_i = 3, \qquad 2r + 1 \leq i \leq n.$$

Now (20) and (21) give

$$\begin{aligned}\sigma_{12} + \ldots + \sigma_{1r} &= (12 - \tau_{12}) + \ldots + (12 - \tau_{1r}) \\ &= 12(r-1) - 2(n + 2r - 5) \\ &= 2(4r - 1 - n);\end{aligned}$$

if the σ_{ij} are all strictly positive, this shows that $n < 4r - 1$. Thus f *is not eutactic if* $n \geqslant 4r - 1$.

If, however, $n \leqslant 4r - 2$ and $r \geqslant 3$, we can show that f is eutactic by taking the particular solution

$$\tau_{lm} = \frac{2(n + 2r - 5)}{r - 1} \qquad (1 \leqslant l < m \leqslant r) \tag{24}$$

of the r equations (21). Then (20) gives

$$\sigma_{jk} = 12 - \frac{2(n + 2r - 5)}{r - 1} = \frac{2(4r - 1 - n)}{r - 1} > 0 \tag{25}$$

for all relevant j, k, and we have exhibited a solution of (17) in which all the coefficients ρ_i, σ_{jk}, τ_{lm} are positive.

We have now established our assertion that f is extreme if and only if (3) holds. In particular, we have shown that the senary form (1), for which $n = 2r = 6$, is extreme.

The form (2) gives some information on the possible structure of perfect, eutactic and extreme forms, as well as extending Coxeter's table (**1**, p. 439) of extreme forms for each $n \geqslant 6$.

Thus Coxeter remarks (**1**, p. 396): "For every known perfect form in less than nine variables there is a solution [of (9)] with the ρ's all equal." However, for the form (2), there is no such solution for any $n \geqslant 2r \geqslant 6$. This assertion is clear if $n > 2r$, from (22) and (23); if $n = 2r$, it follows from (24) and (25), since equality of the τ's and σ's would require

$$4r - 5 = 2r - 1, \ r = 2.$$

Coxeter also remarks (**1**, p. 392): "We do not know whether every perfect form is extreme." The form (2), however, is perfect and non-eutactic (and so not extreme) for any $r \geqslant 3$, $n \geqslant 4r - 1$.

References

1. H. S. M. Coxeter, *Extreme forms*, Can. J. Math. *3* (1951), 391–441.
2. N. Hofreiter, *Ueber Extremformen*, Monatsh. Math. Phys. *40* (1933), 129–152.
3. G. Voronoï, *Sur quelques propriétés des formes quadratiques positives parfaites*, J. reine angew. Math. *133* (1908), 97–178.

University of Sydney, Australia

ON INTEGERS n RELATIVELY PRIME TO $f(n)$

JOACHIM LAMBEK and LEO MOSER

1. Introduction. If m and n are two integers chosen at random, the probability that they are relatively prime **(2**, p. 267**)** is $6\pi^{-2}$. This result may still hold when m and n are functionally related. Thus, Watson **(3)** recently proved that for α irrational, the positive integers n for which $(n, [\alpha n]) = 1$, have density $6\pi^{-2}$. A different proof of a slightly more general result was given by Estermann **(1)**. The present authors found that the number of positive integers not exceeding x, with $(n, [n^{\frac{1}{2}}]) = 1$, is $6\pi^{-2}x + O\,(x^{\frac{1}{2}} \log x)$. In this paper we generalize the latter result. Roughly speaking, *if $f(1), f(2), \ldots$ is a non-decreasing sequence of non-negative integers, tending slowly to infinity, and if the intervals over which $f(m) = n$ increase slowly with n, then the probability that n be relatively prime to $f(n)$ is $6\pi^{-2}$.*

2. Notation. As usual, let $[\alpha]$ denote the largest integer not exceeding α, and let (m, n) be the greatest common divisor of m and n. Small Roman letters usually denote positive integers. Let $f = \{f(1), f(2), \ldots\}$ be any sequence of non-negative integers; then we define as follows:

$Q_f(x)$ is the number of $n \leqslant x$ for which $(n, f(n)) = 1$.

If

$$\lim_{x\to\infty} x^{-1} Q_f(x)$$

exists, we denote it by P_f, and call it *the probability that n and $f(n)$ are relatively prime.*

$R_f\,(x; a, b)$ is the number of multiples of a, not exceeding x, which are mapped onto b by f.

$S_f\,(x; a, b)$ is the number of multiples of a, not exceeding x, which are mapped onto multiples of b by f. Usually, the suffix f in the functions defined above will be omitted.

$f^*(n)$ denotes *the number of m such that $f(m) = n$.*

3. Preliminaries. We assume at the outset that

(i) *f is non-decreasing.*

The following relations between the functions defined are then immediate:

$$S(x; a, b) = \sum_{k=0}^{[b^{-1} f(x)]} R(x; a, kb), \tag{3.1}$$

$$S(x; 1, 1) = x, \tag{3.2}$$

$$f^*(b) = R(\infty; 1, b). \tag{3.3}$$

We now prove several lemmas.

Received July 26, 1954.

LEMMA 1.
$$Q(x) = \sum_{d=1}^{f(x)} \mu(d) \;\; S(x; d, d).$$

Proof. The Möbius function μ has the following property (**2**, p. 234):

$$(3.4) \qquad \sum_{d|n} \mu(d) = 1 \quad (n = 1), \quad \sum_{d|n} \mu(d) = 0 \quad (n > 1).$$

Hence,
$$Q(x) = \sum_{\substack{n<x \\ (n, f(n))=1}} 1 = \sum_{n<x} \sum_{d|(n, f(n))} \mu(d) = \sum_{d=1}^{f(x)} \mu(d) \; S(x; d, d),$$

since $S(x; d, d)$ is the number of $n < x$ such that $d|n$ and $d|f(n)$. The fact that f is non-decreasing ensures that the last summation need not be carried beyond $f(x)$.

LEMMA 2. $|R(x; a, b) - a^{-1} R(x; 1, b)| < 1$.

Proof. By (i), the set of $R(x; 1, b)$ numbers mapped on b by f consists of consecutive integers. The $R(x; a, b)$ multiples of a mapped on b form a subset whose neighboring elements differ by a. The required result now follows from the fact that every set of a consecutive numbers contains exactly one multiple of a, while every set of fewer than a consecutive numbers contains at most one multiple of a.

LEMMA 3. $|S(x; a, b) - a^{-1} S(x; 1, b)| < b^{-1} f(x)$.

Proof. Replace b by kb in Lemma 2 and use (3.1).

We now assume

(ii) *f^* is finite and non-decreasing.*

LEMMA 4.
$$0 < \sum_{k=0}^{s} f^*(bk) - b^{-1} \sum_{k=0}^{sb} f^*(k) < f^*(sb).$$

Proof. The result follows by summing over k the obvious inequalities
$$f^*(bk) > b^{-1}(f^*(bk) + f^*(bk-1) + \ldots + f^*(bk-b+1))$$
and
$$f^*(bk) < b^{-1}(f^*(bk) + f^*(bk+1) + \ldots + f^*(bk+b-1)).$$

LEMMA 5. $S(x; 1, b) - b^{-1} S(x; 1, 1) = O(f^*(f(x)))$.

Proof. We observe that in (3.1) all terms of the sum, with the possible exception of the last, are unaltered by replacing x by ∞. The last term is $O(f^*(f(x)))$. Hence, using (3.3), we have

$$(3.5) \qquad \begin{aligned} S(x; 1, b) &= \sum_{k=0}^{[b^{-1} f(x)]} R(\infty; 1, kb) + O(f^*(f(x))) \\ &= \sum_{k=0}^{[b^{-1} f(x)]} f^*(kb) + O(f^*(f(x))). \end{aligned}$$

Also, putting $b = 1$ in (3.5) and absorbing the last $f(x) - b[b^{-1}f(x)]$ terms in the error term, we obtain

$$b^{-1}S(x;1,1) = b^{-1}\sum_{k=0}^{b[b^{-1}f(x)]} f^*(k) + O(f^*(f(x))). \tag{3.6}$$

The required result now follows from (3.5), (3.6), Lemma 4 with $s = [b^{-1}f(x)]$, and the fact that $f^*(b[b^{-1}f(x)]) < f^*(f(x))$.

4. Results. We proceed to estimate $S(x; a, b)$ and $Q(x)$. Consider the identity

$$\begin{aligned} S(x;a,b) = (ab)^{-1}S(x;1,1) &+ (S(x;a,b) - a^{-1}S(x;1,b)) \\ &+ a^{-1}(S(x;1,b) - b^{-1}S(x;1,1)). \end{aligned} \tag{4.1}$$

Using (3.2) and Lemmas 3 and 5, we obtain from (4.1) that

$$S(x;a,b) = (ab)^{-1}x + O(b^{-1}f(x)) + O(a^{-1}f^*(f(x))). \tag{4.2}$$

It is well known that

$$\sum_{d=1}^{r} d^{-1} = \log r + O(1) \tag{4.3}$$

and

$$\sum_{d=1}^{r} \mu(d)\, d^{-2} = 6\pi^{-2} + O(r^{-1}). \tag{4.4}$$

Using Lemma 1, (4.2) with $a = b = d$, (4.3) and (4.4), we obtain

THEOREM 1. *If* (i) *f is non-decreasing and* (ii) *f^* is finite and non-decreasing, then*

$$Q(x) = 6\pi^{-2}x + O(f(x)\log f(x)) + O(f^*(f(x))\log f(x)) + O(xf(x)^{-1}).$$

Example 1. $f(x) = [x^{\frac12}], f^*(x) = 2x + 1, Q(x) = 6\pi^{-2}x + O(x^{\frac12}\log x)$.

More generally, if k is an integer >1, and $f(x) = [x^{1/k}]$, then

$$Q(x) = 6\pi^{-2}x + O(x^{1-1/k}\log x).$$

An easy consequence of Theorem 1 is

THEOREM 2. *If* (i) *and* (ii) *hold, as well as* (iii) $f(x)\log f(x) = o(x)$ *and* (iv) $f^*(f(x))\log f(x) = o(x)$, *then* $P_f = 6\pi^{-2}$.

Proof. By Theorem 1 and the definition of P_f it suffices to check that $xf(x)^{-1} = o(x)$, that is, $f(x) \to \infty$, and this is a consequence of (i) and (ii).

5. Discussion. Clearly P_f is unaffected by changing the value of $f(x)$ on any set of zero density. Thus one can easily construct functions for which $P_f = 6\pi^{-2}$ but none of (i) to (iv) hold. On the other hand, none of the conditions (i) to (iv) are superfluous for Theorem 2, and they are therefore independent, as may be seen from Examples 2 to 5.

Example 2. $f(2x) = 2[(x/2)^{\frac{1}{2}}]$, $f(2x+1) = f(2x) + 1$. Here $f^*(2x) = f^*(2x + 1) = 2x + 1$. Only (i) is violated. However, since $(n, f(n)) \geqslant 2$ for n even, $Q(x) \leqslant [\frac{1}{2}x]$, and if P exists then $P \leqslant \frac{1}{2} < 6\pi^{-2}$.

Example 3. $f(x) = 2[x^{\frac{1}{2}}]$. Only (ii) is violated, but again $(n, f(n)) \geqslant 2$ for n even, so that $P \neq 6\pi^{-2}$.

Example 4. $f(x) = x$. Only (iii) is violated, but clearly $Q(x) = 1$ and $P = 0$.

Example 5. $f(x) = [\log_{10} x]$. Only (iv) is violated. Let $x = 10^{2r+1}$ and consider all $n = 10^{2r} + 2s < x$, $s \geqslant 1$, so that $(n, f(n)) \geqslant 2$. Their number is $\frac{1}{2}(10^{2r+1} - 10^{2r}) = 0.45x$. Hence $Q(x) < 0.55x$, and $P \neq 6\pi^{-2}$. Actually it is not difficult to see that for this f, P_f does not exist.

References

1. T. Estermann, *On the number of primitive lattice points in a parallelogram*, Can. J. Math., 5 (1953), 456–459.
2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (Oxford, 1938).
3. G. L. Watson, *On integers n relatively prime to* $[\alpha n]$, Can. J. Math., 5 (1953), 451–455.

McGill University *University of Alberta*
Research Institute, Canadian Mathematical Congress

ON SOLUTIONS OF $x^d = 1$ IN SYMMETRIC GROUPS

LEO MOSER AND MAX WYMAN

1. Introduction. Several recent papers have dealt with the number of solutions of $x^d = 1$ in S_n, the symmetric group of degree n. Let us denote this number by $A_{n,d}$ and let $A_{n,2} = T_n$. Chowla, Herstein and Moore **(1)** proved:

1.1 $$T_n = T_{n-1} + (n-1)T_{n-2} \qquad T_0 = T_1 = 1$$

1.2 $$n^{\frac{1}{2}} < T_n/T_{n-1} < n^{\frac{1}{2}} + 1$$

1.3 $$T_n = n!\sum_{j=0}^{[\frac{1}{2}n]} 1/\{2^j j!(n-2j)!\}$$

1.4 $$\sum_{n=0}^{\infty} T_n x^n/n! = e^{x+\frac{1}{2}x^2}$$

1.5 $$T_n \sim \frac{(n/e)^{\frac{1}{2}n} e^{n^{\frac{1}{2}}}}{2^{\frac{1}{2}} e^{1/4}}.$$

In §2 we establish a connection between T_n and the Hermite polynomials. This will enable us to refine 1.5 and to prove a conjecture concerning T_n/T_{n-1} made in **(1)**.

Jacobsthal **(3)** showed that for p a prime

1.6 $$\sum_{n=0}^{\infty} A_{n,p}\, x^n/n! = e^{x+x^p/p}.$$

This was generalized by Chowla, Herstein and Scott **(2)** who proved

1.7 $$\sum_{n=0}^{\infty} A_{n,d}\, x^n/n! = \exp\sum_{k|d} x^k/k.$$

The problem of finding asymptotic formulae for $A_{n,d}$ was proposed in **(2)**. In §3 we obtain asymptotic formulae for T_n and $A_{n,p}$ by a method which also yields results in the general case.

It was pointed out in **(2)** that Frobenius' theorem implies

1.8 $$A_{n,p} \equiv 0 \pmod{p}$$

and this, together with an explicit expression for $A_{n,p}$ constitutes a generalization of Wilson's theorem. Some other arithmetic properties of T_n were developed in **(1)**. In §4 we obtain still another generalization of Wilson's theorem, and also further arithmetic properties of T_n.

In §5 we show how some corresponding results can be obtained for the number of solutions of $x^d = 1$ in alternating groups.

Received June 16, 1954.

2. T_n and the Hermite Polynomials. The recurrence 1.1 is similar to that satisfied by the Hermite polynomials, namely

2.1 $$H_n(x) = 2x\,H_{n-1}(x) - 2(n-1)\,H_{n-2}(x), \quad H_0(x) = 1, \quad H_1(x) = 2x.$$

In fact, we prove the following theorem:

2.2 $$T_n = \frac{H_n(i/2^{\frac{1}{2}})}{i^n 2^{\frac{1}{2}n}}, \qquad i^2 = -1.$$

Since 2.2 is readily verified for $n = 0$ and $n = 1$ we may proceed by induction. Using the induction hypothesis and 2.1 we obtain,

$$T_{n-1} + (n-1)T_{n-2}$$
$$= \frac{H_{n-1}(i/2^{\frac{1}{2}})}{i^{n-1}2^{\frac{1}{2}(n-1)}} + \frac{(n-1)H_{n-2}(i/2^{\frac{1}{2}})}{i^{n-2}2^{\frac{1}{2}(n-2)}} = \frac{H_n(i/2^{\frac{1}{2}})}{i^n 2^{\frac{1}{2}n}}$$

which completes the proof.

From 2.2 we can obtain the asymptotic expansion of T_n. The asymptotic expansion of $H_n(x)$ is given by Szegö (4, p. 194) to be

2.3 $$\lambda_n^{-1}e^{-\frac{1}{2}x^2}H_n(x) = \cos(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)\sum_{\nu=0}^{p-1} U_\nu(x)\,N^{-\nu}$$
$$+ N^{-\frac{1}{2}}\sin(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)\sum_{\nu=0}^{p-1} V_\nu(x)\,N^{-\nu} + \exp\{-N^{\frac{1}{2}}|\mathscr{I}(x)|\}\,O(n^{-p}),$$

where $N = 2n + 1$ and

$$\lambda_n = \frac{\Gamma(n+1)}{\Gamma(\frac{1}{2}n+1)} \text{ or } \lambda_n = \frac{\Gamma(n+2)}{\Gamma(\frac{1}{2}n+3/2)}N^{-\frac{1}{2}}$$

according as n is even or odd. The coefficients $U_\nu(x)$ and $V_\nu(x)$ are polynomials depending on ν; they contain only even and odd powers of x, respectively. The first two terms of this expansion yield

2.4 $$H_n(x) \sim \frac{\Gamma(n+1)}{\Gamma(\frac{1}{2}n+1)}e^{\frac{1}{2}x^2}\left(\cos(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi) + \tfrac{1}{6}x^3N^{-\frac{1}{2}}\sin(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi)\right).$$

Using 2.2, 2.4 and the asymptotic expansion of the gamma function we obtain the theorem

2.5 $$T_n \sim \frac{(n/e)^{\frac{1}{2}n}e^{n^{\frac{1}{2}}}}{2^{\frac{1}{2}}e^{1/4}}\left(1 + \frac{7}{24n^{\frac{1}{2}}} + \dots\right).$$

This is a refinement of 1.5. A numerical check of 2.5 and a still more accurate formula for T_n derived in §3 will be given later in the paper.

We next consider $R_n = T_n/T_{n-1}$. By 1.1 we have

2.6 $$R_{n+1} = 1 + n/R_n.$$

Iterating 2.6 gives the continued fraction expansion

2.7 $$R_{n+1} = 1 + \frac{n}{1+}\ \frac{n-1}{1+}\ \frac{n-2}{1+}\cdots\frac{1}{1}.$$

We now use elementary means to sharpen the bounds for R_n given in 1.2. Let α_n be the positive solution of $x^2 - x - n = 0$, i.e.,

2.8 $$\alpha_n = (1 + \tfrac{1}{2}(4n + 1)^{\frac{1}{2}}).$$

We prove

2.9 $$\alpha_{n-1} < R_n < \alpha_n.$$

Proof. We proceed by induction over n. The result is trivial for $n = 1$. Assume it true for $n = K$, then

$$R_{K+1} = 1 + K/R_K > 1 + K/\alpha_K = \alpha_K.$$

Also,

$$R_{K+1} = 1 + K/R_K < 1 + K/\alpha_{K-1}.$$

It remains to show that

$$1 + K/\alpha_{K-1} < \alpha_{K+1}$$

and this follows from 2.8 and simple algebraic manipulation. An easy consequence of 2.9 is

2.10 $$\lim_{n\to\infty} (R_n - n^{\frac{1}{2}}) = \tfrac{1}{2}.$$

In **(1)** it was conjectured that

2.11 $$R_n \sim n^{\frac{1}{2}} + A + Bn^{-\frac{1}{2}} + Cn^{-1} + \ldots$$

for appropriate constants A, B, $C, \ldots$. We shall prove this conjecture and obtain the following theorem

2.12 $$R_n \sim n^{\frac{1}{2}} + \tfrac{1}{2} - \tfrac{1}{8}n^{-\frac{1}{2}} + \ldots.$$

Proof. Let us consider

2.13 $$f_n(x) = xH_n(x)/H_{n-1}(x).$$

From 2.2 and 2.13 we have

2.14 $$R_n = -f_n(i/2^{\frac{1}{2}}).$$

However, it is well known that

2.15 $$H_n'(x) = 2n\, H_{n-1}(x).$$

Hence,

2.16 $$f_n(x) = 2n\, x \frac{H_n(x)}{H_n'(x)} = \frac{2nx}{(\log H_n(x))'}.$$

If we restrict x to be pure imaginary, say $x = it$, $t > 0$, then the expansion 2.3 takes the form

2.17 $$H_n(x) \sim \tfrac{1}{2}\lambda_n e^{\frac{1}{2}x^2} i^n e^{-iN^{\frac{1}{2}}x}\left[\sum_{\nu=0}^{p-1} U_\nu(x)\, N^{-\nu} + \frac{1}{iN^{\frac{1}{2}}}\sum_{\nu=0}^{p-1} V_\nu(x)\, N^{-\nu}\right].$$

From 2.15, $H_n'(x)$ is known to have an asymptotic expansion of the form 2.3. Hence differentiation of 2.3 is justified and leads to

2.18 $$(\log H_n(x))' \sim x - iN^{\frac{1}{2}} + \sum_{\nu=1}^{p-1} q_\nu(x)\, N^{-\nu} + \frac{1}{iN^{\frac{1}{2}}}\sum_{\nu=0}^{p-1} Q_\nu(x)\, N^{-\nu}$$

where again, $q_\nu(x)$ and $Q_\nu(x)$ are polynomials in x. Since these polynomials contain only even powers of x, and odd powers of x respectively, the coefficients of the various powers of $N^{-\nu}$ will be real in 2.17 and pure imaginary in 2.18. Hence for $x = i/2^{\frac{1}{2}}$,

2.19 $$f_n(i/2^{\frac{1}{2}}) \sim \frac{2^{\frac{1}{2}} n}{2^{\frac{1}{2}} - N^{\frac{1}{2}} + \beta_1 + \beta_2 N^{-\frac{1}{2}} + \ldots}$$

where $\beta_1, \beta_2 \ldots$ are constants. Hence,

2.20 $$R_n \sim \frac{2^{\frac{1}{2}} n}{N^{\frac{1}{2}}}\left(1 + \frac{\gamma_1}{N^{\frac{1}{2}}} + \frac{\gamma_2}{N} + \ldots\right)$$

where $\gamma_1, \gamma_2 \ldots$ are constants. Since $N = 2n+1$ we may expand in terms of n and find

2.21 $$R_n \sim n^{\frac{1}{2}}\left(1 + \frac{\delta_1}{n^{\frac{1}{2}}} + \frac{\delta_2}{n} + \ldots\right)$$

where $\delta_1, \delta_2 \ldots$ are constants. We have calculated the first three terms to be those given in 2.12.

3. Asymptotic Expansions. Previously in this paper we obtained an asymptotic expansion of T_n by recognizing the relationship between T_n and the Hermite polynomials. This method of course does not help us when considering the more general problem involving $A_{n,d}$. In this section we shall obtain an asymptotic expansion for T_n by a different method. This method not only applies to $A_{n,d}$, but also to many other problems, some of which will be discussed in a later paper.

In our method we make use of a result, indicated in the following lemma, which is probably known. Since we have been unable to find a reference we include, for completeness, a proof of the

LEMMA. *Let $f(z)$ be a function of a complex variable z regular in a neighborhood of $z = 0$. If*
(a) $f(0) = 0$,
(b) *the Maclaurin expansions of $f(z)$, $e^{f(z)}$ are*

$$f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad e^{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

(c) $|a_k| < K\sigma^k$,
where K, σ are positive numbers, then

3.1 $$|b_k| < K\,\sigma^k(1+K)^{k-1}.$$

Proof. By Taylor's Theorem,

3.2 $$a_k = \frac{1}{k!}\left(\frac{d^k f(z)}{dz^k}\right)_{z=0}$$

3.3 $$b_k = \frac{1}{k!}\left(\frac{d^k e^{f(z)}}{dz^k}\right)_{z=0}.$$

Since $b_1 = a_1$ the result is trivially true for $k = 1$. From 3.3 we may write

$$b_{k+1} = \frac{1}{(k+1)!}\frac{d^k}{dz^k}\left[e^{f(z)}\frac{df}{dz}\right]_{z=0}. \tag{3.4}$$

Expanding the kth derivative of a product we have

$$b_{k+1} = \frac{1}{(k+1)}\sum_{s=0}^{k} b_s a_{k-s+1}(k - s + 1), \qquad b_0 = 1. \tag{3.5}$$

Hence

$$|b_{k+1}| < \sum_{s=0}^{k} |b_s||a_{k-s+1}|. \tag{3.6}$$

From (c) we obtain

$$|b_{k+1}| < K\,\sigma^{k+1}\left(1 + \sum_{s=1}^{k} |b_s|\,\sigma^{-s}\right). \tag{3.7}$$

From 3.7 the result follows easily by induction.

We now proceed to outline our method for finding an asymptotic expansion of T_n. By 1.4 and Cauchy's theorem,

$$T_n = \frac{n!}{2\pi i}\int_C (e^{z+\frac{1}{2}z^2})\, z^{-(n+1)}dz, \tag{3.8}$$

where C is the circle $z = Re^{i\theta}$. Hence T_n can be expressed as

$$T_n = H\int_{-\pi}^{\pi} e^{f(\theta)}d\theta, \tag{3.9}$$

where

$$H = (n!\, e^{R+\frac{1}{2}R^2})/2\pi R^n \tag{3.10}$$

and

$$f(\theta) = R(e^{i\theta} - 1) + \tfrac{1}{2}R^2(e^{2i\theta} - 1) - in\theta. \tag{3.11}$$

If we let

$$\epsilon = R^{-3/4} \tag{3.12}$$

and

$$I = \int_{\epsilon}^{\pi} e^{f(\theta)}d\theta \tag{3.13}$$

then a simple calculation shows that $|I| = O(e^{-R^{\frac{1}{2}}})$. Since we shall show that our asymptotic expansion of the integral in 3.9 involves only powers of $1/R$ we may drop 3.13 and write

$$T_n \sim H\int_{-\epsilon}^{\epsilon} e^{f(\theta)}d\theta. \tag{3.14}$$

Expanding $f(\theta)$ in a Maclaurin expansion we obtain

$$f(\theta) = i(R^2 + R - n)\theta - \theta^2(R^2 + \tfrac{1}{2}R) + \sum_{k=3}^{\infty} (R + 2^{k-1}R^2)\frac{(i\theta)^k}{k!}. \tag{3.15}$$

We now choose R so that

$$R^2 + R - n = 0 \tag{3.16}$$

and define ϕ by means of

$$\phi = \theta(R^2 + \tfrac{1}{2}R)^{\frac{1}{2}}. \tag{3.17}$$

We may then write 3.14 in the form

3.18 $$T_n \sim J \int_{-c}^{c} e^{-\phi^2 + F(z,\phi)} d\phi$$

where

3.19 $$c = \epsilon(R^2 + \tfrac{1}{2}R)^{\frac{1}{2}}$$

3.20 $$z = 1/R$$

3.21 $$J = H/(R^2 + \tfrac{1}{2}R)^{\frac{1}{2}}$$

3.22 $$F(z, \phi) = \sum_{k=3}^{\infty} (z + 2^{k-1})(1 + \tfrac{1}{2}z)^{-\frac{1}{2}k} z^{k-2} \frac{(i\phi)^k}{k!}.$$

Since $\epsilon = R^{-3/4}$, $c = O(R^{1/4})$ and $c \to \infty$ as $R \to \infty$. Further for any fixed ϕ, $F(z, \phi)$ is regular in the neighborhood $|z| < 2$ and $e^{F(z,\phi)}$ will have a Maclaurin expansion of the form

3.23 $$e^F = \sum_{m=0}^{\infty} \psi_m(\phi)\, z^m,$$

where $\psi_m(\phi)$ is a polynomial in ϕ. Hence

3.24 $$T_n \sim J\left[\sum_{m=0}^{s-1} \left(\int_{-c}^{c} e^{-\phi^2} \psi_m(\phi) d\,\phi\right) z^m + R_s\right]$$

where

3.25 $$R_s = \int_{-c}^{c} \left(e^{-\phi^2} \sum_{m=s}^{\infty} \psi_m(\phi) z^m\right) d\phi.$$

Since $\psi_m(\phi)$ is a polynomial in ϕ and $c = O(R^{1/4})$ one can place

3.26 $$\int_{-c}^{c} e^{-\phi^2} \psi_m(\phi)\, d\phi = \int_{-\infty}^{\infty} e^{-\phi^2} \psi_m(\phi)\, d\phi$$

with an error that is of exponential order. Hence we write

3.27 $$T_n \sim J\left[\sum_{m=0}^{s-1} \left(\int_{-\infty}^{\infty} e^{-\phi^2} \psi_m(\phi)\, d\phi\right) z^m + R_s\right].$$

In order to complete our proof we merely have to show that, for fixed s, $|R_s| = 0(|z|^s)$.

If the Maclaurin expansion of $F(z, \phi)$, as a function of z, is written

3.28 $$F(z, \phi) = \sum_{r=1}^{\infty} a_r(\phi)\, z^r,$$

then by 3.22

3.29 $$a_r(\phi) = \frac{1}{r!} \sum_{k=3}^{r+2} \left[\frac{d^r}{dz^r} (z + 2^{k-1})(1 + \tfrac{1}{2} z)^{-\frac{1}{2}k} z^{k-2}\right]_{z=0} \frac{(i\phi)^k}{k!}.$$

By using Cauchy's theorem for derivatives one may easily show, for $|z|<1$, that

3.30 $$\left|\frac{1}{r!} \frac{d^r}{dz^r} (z + 2^{k-1})(1 + \tfrac{1}{2}z)^{-\frac{1}{2}k} z^{k-2}\right|_{z=0} < 2^{2k}.$$

Hence

3.31 $$|a_r(\phi)| < \sum_{k=3}^{r+2} \frac{(4|\phi|)^k}{k!}.$$

From 3.31 it is easy to show by induction that

3.32 $$|a_r(\phi)| < (4|\phi|^2(1 + 4|\phi|)^r.$$

Making use of our lemma with $K = (4|\phi|)^2$, $\sigma = 1 + 4|\phi|$ we have

3.33 $$|\psi_m(\phi)| < (4|\phi|)^2(1 + 4|\phi|)^m(2 + (4|\phi|)^2)^{m-1}.$$

Hence

3.34 $$\left|\sum_{m=s}^{\infty} \psi_m(\phi)\, z^m\right| < \frac{(4|\phi|)^2(1 + 4|\phi|)^s(2 + (4|\phi|)^2)^{s-1}|z|^s}{M}$$

where

3.35 $$M = 1 - (1 + 4|\phi|)(2 + (4|\phi|)^2)|z|.$$

Now $z = 1/R$ and in 3.25 $|\phi| < c = O(R^{1/4})$. Therefore $|z|\,|\phi|^3 = O(R^{-1/4}) \to 0$ as $R \to \infty$. Hence for sufficiently large R, $M > \frac{1}{2}$. Thus one can say

3.36 $$\left|\sum_{m=s}^{\infty} \psi_m(\phi)\, z^m\right| < P_s(|\phi|)|z|^s,$$

where $P_s(|\phi|)$ is a polynomial in $|\phi|$. From 3.25

3.37 $$|R_s| < \left(\int_{-c}^{c} e^{-\phi^2} P_s(|\phi|)\, d\phi\right)|z|^s < \int_{-\infty}^{\infty} e^{-\phi^2} P_s(|\phi|)\, d\phi |z|^s.$$

Since the integral exists for each fixed s we must have

3.38 $$|R_s| = O(|z|^s)$$

for a fixed s. This completes the proof of the theorem. Hence, an asymptotic expansion of T_n is given by

3.39 $$T_n \sim \frac{n!\, e^{R+\frac{1}{2}R^2}}{2\pi R^n (R^2 + \frac{1}{2}R)^{\frac{1}{2}}}\left[\sum_{m=0}^{\infty} \frac{\int_{-\infty}^{\infty} e^{-\phi^2} \psi_m(\phi)\, d\phi}{R^m}\right]$$

where $R^2 + R - n = 0$.

We have used 3.39 to show that up to and including terms of the order $1/n$

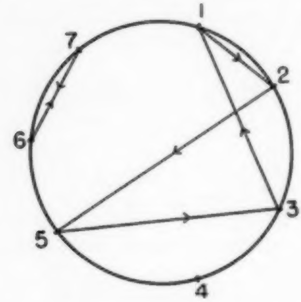
3.40 $$T_n \sim \frac{1}{2^{\frac{1}{2}} e^{1/4}}\left(\frac{n}{e}\right)^{\frac{1}{2}n} e^{n^{\frac{1}{2}}}\left(1 + \frac{7}{24\sqrt{n}} - \frac{119}{1152n} + \dots\right)$$

Numerically T_{100} is given by 3.39 to be $2.40537 \ldots \times 10^{82}$. Correct to six significant figures $T_{100} = 2.40533 \ldots \times 10^{82}$.

The method outlined here can be easily generalized, and can be used to find an asymptotic expansion for $A_{n,d}$. For example, the first term of the asymptotic expansion of $A_{n,p}$ is given by

3.41 $$A_{n,p} \sim p^{-\frac{1}{2}}\left(\frac{n}{e}\right)^{n(1-1/p)} e^{n^{1/p}}, \qquad p > 2.$$

4. Arithmetic properties of $A_{n,d}$. Let a permutation x, on n letters, be represented as a product of disjoint cycles. It will also be convenient to represent x by a diagram in the following way: Let a unit circle have n equi-spaced points on its circumference, labelled $1, 2, \ldots, n$. If x takes i into j, join i and j by a directed line segment. Thus a permutation x will be represented by a circle and a set of directed inscribed polygons. For example, the permutation (1253) (4)(67) corresponds to Figure 1.



A rotation of the circle through $2\pi m/n$ ($m = 1, 2, \ldots n - 1$) leaving the labels fixed will, in general, yield a new permutation having the same order as the old one. If $n = p$, p a prime, the only diagrams left unaltered by a rotation of the form mentioned above will be those corresponding to either the unit permutation I, or the regular directed p-gons, of which there are $p - 1$. We now use these concepts to prove

4.1 $$A_{p,d} \equiv 1 \pmod p), d \neq p, \qquad A_{p,p} \equiv 0 \pmod p).$$

Since $A_{p,p}$ is easily seen to be $(p-1)! + 1$, the result may be viewed as still another generalization of Wilson's theorem.

Proof. Suppose $x^d = 1$, $d \neq p$. Apart from $x = I$, all solutions (diagrams) must come in sets of p by rotation through $2\pi m/p$, $m = 0, 1, \ldots, p - 1$. This proves the first part of 4.1. If $x^p = 1$, then $x = I$ or the diagram for x must consist of a directed p-gon (of which there are $(p - 1)!$). If we eliminate the diagram for I, and the $p - 1$ directed regular p-gons, then the remaining $A_{p,p} - p$ diagrams must again come in sets of p, by rotation. Thus the proof is complete.

In **(1)** it was shown that

4.2 $$T_{n+m} \equiv T_n \pmod m \qquad m \text{ odd}.$$

We next derive a similar theorem in which m is unrestricted, namely

4.3 $$T_{n+m} \equiv T_n \,.\, T_m \pmod m.$$

Proof. For $n = 0$ the theorem is trivial and for $n = 1$ it follows immediately from 1.1. Assuming it true for $n < k$ we have

$$\begin{aligned} T_{k+m} &= T_{k+m-1} + (k + m - 1)T_{k+m-2} \\ &\equiv T_{k-1} \cdot T_m + (k - 1)\, T_{k-2} \cdot T_m = T_k \cdot T_m \pmod{m}. \end{aligned}$$

5. **The alternating group.** Let $B_{n,4}$ denote the number of solutions of $x^4 = 1$ in the alternating group on n letters. Further, let $B_{n,2} = U_n$. Define V_n and W_n by

5.1 $$U_n + V_n = T_n, \quad U_n - V_n = W_n.$$

To study U_n it clearly suffices to consider W_n. The analogue of 1.1 is given by

5.2 $$W_n = W_{n-1} - (n - 1)W_{n-2}, \qquad W_0 = W_1 = 1.$$

Proof. The only elements of order two in the alternating group on n letters are those which are the product of an even number of disjoint transpositions and the unit element. Hence the number of even elements of order two which can be obtained from the permutations of the digits $1, 2, \ldots, n - 1$ alone is U_{n-1}. Further, since a single transposition is odd, the only other such elements are obtained by involving the digit n in a transposition with some other digit, and multiplying by any other *odd* permutation of order two, involving the remaining $n - 2$ digits. Their number is clearly $(n - 1)\, V_{n-2}$. Thus

5.3 $$U_n = U_{n-1} + (n - 1)\, V_{n-2}.$$

Similarly we obtain

5.4 $$V_n = V_{n-1} + (n - 1)\, U_{n-2}.$$

Subtracting 5.4 from 5.3 yields 5.2.

Following the lines of the proof of 1.3 given in **(1)** we easily obtain

5.5 $$\sum_{n=0}^{\infty} W_n x^n/n! = e^{x-\frac{1}{2}x^2}.$$

The analogue of 2.2 is given by

5.6 $$W_n = \frac{H_n(1/2^{\frac{1}{2}})}{2^{\frac{1}{2}n}}.$$

Since the proof is essentially the same as that of 2.2 we omit it. The well-known explicit formula for $H_n(x)$ yields the following analogue of 1.4:

5.7 $$W_n = n! \sum_{j=0}^{[\frac{1}{2}n]} (-1)^j/2^j j!(n - 2j)!.$$

Combining 1.3 and 5.7 yields

5.8 $$U_n = n! \sum_{k=0}^{[n/4]} 1/\{4^k(2k)!(n - 4k)!\}.$$

From the asymptotic formula for $H_n(x)$ and 5.6 we obtain

5.9 $$W_n \sim (n/e)^{\frac{1}{2}n} 2^{\frac{1}{2}} e^{1/4} \cos \{(n+\tfrac{1}{2})^{\frac{1}{2}} - \tfrac{1}{2}n\pi\}.$$

The arithmetic properties of T_n and $A_{n,d}$ also have direct analogues. Thus we obtain without difficulty,

5.10 $$B_{p,d} \equiv 1 \pmod{p},\ d \neq p, \qquad B_{p,p} = A_{p,p}.$$

Finally, the analogue of 4.3 is given by

5.11 $$W_{n+m} \equiv W_n \, . \, W_m \pmod{m}.$$

The following is a short table of T_n, U_n, W_n.

n	T_n	U_n	W_n
0	1	1	1
1	1	1	1
2	2	1	0
3	4	1	−2
4	10	4	−2
5	26	16	6
6	76	46	16
7	232	106	−20
8	764	281	−132
9	2620	1324	28
10	9496	5356	1216

In concluding we wish to thank F. L. Miksa for computing for us the exact values of T_n up to and including T_{152}. These values have been of value to us in checking different forms of our asymptotic formulae.

References

1. S. Chowla, I. N. Herstein, and K. Moore, *On recursions connected with symmetric groups* I, Can. J. Math. *3* (1951), 328–334.
2. S. Chowla, I. N. Herstein, and W. R. Scott, *The solutions of $x^d = 1$ in symmetric groups*, Norske Vid. Selsk., *25* (1952), 29–31.
3. E. Jacobsthal, *Sur le nombre d'elements du group symetrique S_n dont l'ordre est un nombre premier*, Norske Vid. Selsk., *21* (1949), 49–51.
4. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. publications (New York, 1939).

Research Institute, Canadian Mathematical Congress
and
University of Alberta

THE GROUP RING OF A CLASS OF INFINITE NILPOTENT GROUPS

S. A. JENNINGS

Introduction. In this paper we study the (discrete) group ring Γ of a finitely generated torsion free nilpotent group $\mathfrak{G}$ over a field of characteristic zero. We show that if Δ is the ideal of Γ spanned by all elements of the form $G - 1$, where $G \in \mathfrak{G}$, then

$$\Delta \supset \Delta^2 \supset \Delta^3 \supset \ldots \supset \Delta^w \supset \Delta^{w+1} \supset \ldots$$

and the only element belonging to Δ^w for all w is the zero element (c.f. (4.3) below). This fact enables us to topologize Γ in a natural way. We may then define for $\mathfrak{G}$ "dimensional subgroups" relative to the ideal Δ which are the analogues of those considered by Magnus **(7; 8; 9)** for a free group. In part II we associate a Lie algebra with $\mathfrak{G}$ in a natural manner via the group ring, and show that $\mathfrak{G}$ is a subgroup of the simply connected Lie group determined by this Lie algebra. To some extent these last results overlap those of Malcev **(11; 11)**. However, our approach differs greatly from his, since our methods are intrinsic in the sense that topological considerations from the theory of Lie groups do not intervene.

Part I: The Group Ring

1. Preliminary notions. We follow the notation in **(1)** and write

$$(H, K) = H^{-1}K^{-1}HK$$

for the commutator of elements H, K of a group $\mathfrak{G}$, while if $\mathfrak{H}$, $\mathfrak{K}$ are subgroups of $\mathfrak{G}$ then $(\mathfrak{H}, \mathfrak{K})$ is the subgroup of $\mathfrak{G}$ generated by all (H, K) with $H \in \mathfrak{H}$, $K \in \mathfrak{K}$, and similarly for higher commutators. The group $\mathfrak{G}$ is *nilpotent and of class c* if the lower central series of G is as follows:

(1.0.1) $$\mathfrak{G} = \mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \ldots \supset \mathfrak{G}_c \supset \mathfrak{G}_{c+1} = \{1\}$$

where $\mathfrak{G}_{i+1} = (\mathfrak{G}_i, \mathfrak{G})$ $(i = 1, 2, \ldots c)$. A series of normal subgroups

(1.0.2) $$\mathfrak{G} = \mathfrak{K}_1 \supseteq \mathfrak{K}_2 \supseteq \ldots \supseteq \mathfrak{K}_m \supseteq \mathfrak{K}_{m+1} = \{1\}$$

is a *central series* if $(\mathfrak{K}_i, \mathfrak{G}) \subseteq \mathfrak{K}_{i+1}$ $(i = 1, 2, \ldots m)$ and the existence of a

Received July 29, 1953. Part I of this paper was presented to the American Mathematical Society, August 23, 1946. This paper was completed while the author was a Fellow at the Summer Research Institute of the Canadian Mathematical Congress, 1952 and 1953. The author wishes to express his thanks to the Board of Governors of the University of British Columbia for a research grant in connection with this investigation.

central series is necessary and sufficient for a group to be nilpotent. We recall also that for any central series we have

$$(1.0.3) \qquad (\mathfrak{K}_j, \mathfrak{G}_i) \subseteq \mathfrak{K}_{i+j}, \qquad i, j = 1, 2, \ldots .$$

The upper central series of a nilpotent group G:

$$(1.0.4) \qquad \mathfrak{G} = \mathfrak{Z}_c \supset \mathfrak{Z}_{c-1} \supset \ldots \supset \mathfrak{Z}_1 \supset \mathfrak{Z}_0 = \{1\}$$

is usually defined by taking $\mathfrak{Z}_1$, as the centre of $\mathfrak{G}$, and for $i > 1$, $\mathfrak{Z}_i$ as the subgroup in $\mathfrak{G}$ corresponding to the centre of $\mathfrak{G}/\mathfrak{Z}_{i-1}$ in the homomorphism $\mathfrak{G} \to \mathfrak{G}/\mathfrak{Z}_{i-1}$. The following equivalent definition of the series (1.0.4) is probably well known:

THEOREM 1.1. *$\mathfrak{Z}_i$ is the largest normal subgroup of $\mathfrak{G}$ such that*

$$(\mathfrak{Z}_i, \mathfrak{G}, \mathfrak{G}, \ldots \mathfrak{G}) = 1,$$

where the number of $\mathfrak{G}$'s in the above is i.

Proof. Since $\mathfrak{Z}_1$ may be defined as the largest normal subgroup for which $(\mathfrak{Z}_1, \mathfrak{G}) = 1$, assume the theorem true for $\mathfrak{Z}_{i-1}$, and let $\mathfrak{N}$ be any normal subgroup of $\mathfrak{G}$ such that

$$(\mathfrak{N}, \mathfrak{G}, \mathfrak{G}, \ldots \mathfrak{G}) = 1 \qquad \text{for } i \text{ factors } \mathfrak{G}.$$

Then

$$((\mathfrak{N}, \mathfrak{G}), \mathfrak{G}, \ldots \mathfrak{G}) = 1$$

and $(\mathfrak{N}, \mathfrak{G}) \subseteq \mathfrak{Z}_{i-1}$ by induction. That is, $\mathfrak{N}$ is in the centre of $\mathfrak{G}$ modulo $\mathfrak{Z}_{i-1}$ and hence $\mathfrak{N} \subseteq \mathfrak{Z}_i$, from which it follows that $\mathfrak{Z}_i$ is the maximal normal subgroup having the property $(\mathfrak{N}, \mathfrak{G}, \ldots \mathfrak{G}) = 1$ with i factors $\mathfrak{G}$.

A group will be said to be *torsion free* if every element of the group $\neq 1$ is of infinite order. Using (1.1) we now establish, in slightly more general form, a result due to Malcev (**9**, corollary 2).

THEOREM 1.2. *Let $\mathfrak{G}$ be a torsion free nilpotent group with upper central series*

$$\mathfrak{G} = \mathfrak{Z}_c \supset \mathfrak{Z}_{c-1} \supset \ldots \supset \mathfrak{Z}_1 \supset \mathfrak{Z}_0 = \{1\}.$$

Then $\mathfrak{Z}_i/\mathfrak{Z}_{i-1}$ is torsion free.

Proof. Suppose $G \in \mathfrak{Z}_i$ and $G \notin \mathfrak{Z}_{i-1}$, but $G^\alpha \in \mathfrak{Z}_{i-1}$ for some positive integer α: such a G certainly exists if $\mathfrak{Z}_i/\mathfrak{Z}_{i-1}$ is not torsion free. Then

$$(G^\alpha, G_1) \in \mathfrak{Z}_{i-2},$$

for all $G_1 \in \mathfrak{G}$, and because of the identity

$$(PQ, R) = (P, R)(P, R, Q)(Q, R)$$

we have

$$(G^\alpha, G_1) \equiv (G, G_1)^\alpha \equiv 1 \pmod{\mathfrak{Z}_{i-2}}.$$

Using $(G, G_1)^n$ instead of G^n we show similarly that

$$(G^n, G_1, G_2) = (G, G_1, G_2)^n \equiv 1 \pmod{\mathfrak{Z}_{i-3}},$$

and finally

$$(G, G_1, G_2, \ldots, G_{i-1})^n \equiv 1 \pmod{\mathfrak{Z}_0},$$

for all $G_1, \ldots, G_{i-1} \in \mathfrak{G}$, or, since $\mathfrak{Z}_0 = \{1\}$ and $\mathfrak{G}$ is torsion free,

$$(G, G_1, G_2, \ldots G_{i-1}) = 1$$

for all $G_1, \ldots G_{i-1} \in \mathfrak{G}$. But by (1.1) this implies that $G \in \mathfrak{Z}_{i-1}$, which is a contradiction, so that our result is established.

COROLLARY 1.3. *If $\mathfrak{G}$ is a torsion free nilpotent group, then $\mathfrak{G}/\mathfrak{Z}_i$ $(i = 1, 2, \ldots c - 1)$ is torsion free.*

2. Finitely generated torsion free nilpotent groups. Infinite solvable groups with maximal condition for subgroups have been investigated by Hirsch (**3**; **4**; **5**), who calls such groups "*S*-groups." We show first that every finitely generated nilpotent (and therefore a fortiori solvable) group is an *S*-group.

THEOREM 2.1. *Every finitely generated nilpotent group satisfies the maximal condition for subgroups.*

Proof. Let $\mathfrak{G}$ be nilpotent and finitely generated by the elements $P_1, P_2, \ldots P_r$, and let the lower central series of $\mathfrak{G}$ be as in (1.0.1): then by (**1**, Theorem 2.81),

$$\mathfrak{G}_w = \{Q_1, Q_2, \ldots Q_s, \mathfrak{G}_{w+1}\},$$

where $Q_1, Q_2, \ldots, Q_s$ are the various formally distinct commutators of weight w in $P_1, P_2, \ldots, P_r$, there being only a finite number of the Q's since there are only a finite number of the P's. Hence $\mathfrak{G}_w/\mathfrak{G}_{w+1}$ is an abelian group with a finite number of generators, and the series (1.0.1) satisfies the condition (**3**, (1.11)), so that $\mathfrak{G}$ is an *S*-group, as required.

We recall (**4**, Theorem 2.22) that the elements of a finitely generated nilpotent group $\mathfrak{G}$ which are of finite order form a normal subgroup $\mathfrak{F}$, so that the quotient group $\mathfrak{G}/\mathfrak{F}$ is torsion free. In what follows we call any group which is torsion free, nilpotent and finitely generated an *N-group*. That such groups exist follows from the above remark.

Now Hirsch has shown also (**4**, Theorem 2.311) that in any finitely generated nilpotent group $\mathfrak{G}$ there exist series of subgroups, each normal in $\mathfrak{G}$:

(2.1.1) $$\mathfrak{G} = \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \mathfrak{A}_3 \supset \ldots \supset \mathfrak{A}_r \supset \mathfrak{A}_{r+1} = (1)$$

with the properties

(2.1.2) $\mathfrak{A}_i/\mathfrak{A}_{i+1}$ is either cyclic of prime order, or an infinite cyclic group.

(2.1.3) The number of infinite cyclic factors in any series (2.1.1) satisfying (2.1.2) is an invariant of the group $\mathfrak{G}$. (**4**, Theorem 2.23).

We prove now that if $\mathfrak{G}$ is an N-group, we may find series (2.1.1) all of whose factors are infinite cyclic, and indeed such that the series itself is a central series:

THEOREM 2.2. *Any N-group $\mathfrak{G}$ has at least one central series*

(2.2.0) $$\mathfrak{G} = \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \ldots \supset \mathfrak{F}_r \supset \mathfrak{F}_{r+1} = \{1\},$$

such that

(1) $\mathfrak{F}_i/\mathfrak{F}_{i+1}$ $(i = 1, 2, \ldots r)$ *is an infinite cyclic group,*

(2) $(\mathfrak{F}_i, \mathfrak{G}) \subseteq \mathfrak{F}_{i+1}$.

The length r of any such series is an invariant of the group, which we call the rank of $\mathfrak{G}$.

Proof. The invariance of the rank will follow from (2.1.3). Now by (1.2), since $\mathfrak{G}$ is an N-group, the factors $\mathfrak{Z}_i/\mathfrak{Z}_{i-1}$ of the upper central series (1.0.4) of $\mathfrak{G}$ are torsion free, and by (**3**, Theorem 1.33), are finitely generated, so that $\mathfrak{Z}_i/\mathfrak{Z}_{i-1}$ is a direct product of a finite number of infinite cyclic groups. We may therefore refine the upper central series of $\mathfrak{G}$ so that between any two consecutive terms $\mathfrak{Z}_i$ and $\mathfrak{Z}_{i-1}$ we have a finite chain of subgroups

$$\mathfrak{Z}_i \supset \mathfrak{U}_{i1} \supset \mathfrak{U}_{i2} \supset \ldots \supset \mathfrak{U}_{is_i} = \mathfrak{Z}_{i-1},$$

so that each factor is infinite cyclic. Since

$$(\mathfrak{Z}_i, \mathfrak{G}) \subseteq \mathfrak{Z}_{i-1}, \quad (\mathfrak{U}_{ij}, \mathfrak{G}) \subseteq \mathfrak{Z}_{i-1} \subseteq \mathfrak{U}_{i,j+1},$$

and hence the refinement forms part of a central series of $\mathfrak{G}$.

Any series satisfying (1) and (2) of (2.2) will be called an $\mathfrak{F}$-*series* of the N-group $\mathfrak{G}$. It can be readily verified that a group is an N-group if and only if it has $\mathfrak{F}$-series.

The following follows at once from (**4**, Theorem 2.312)

THEOREM 2.3. *Let $\mathfrak{G}$ be an N-group of rank r, and let $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{H}$ is an N-group of rank s. Then $\mathfrak{H}$ is of rank $r - s$, and there is an $\mathfrak{F}$-series of $\mathfrak{G}$ such that $\mathfrak{F}_{s+1} = \mathfrak{H}$.*

COROLLARY 2.4. *If $\mathfrak{G}$ has an $\mathfrak{F}$-series*

$$\mathfrak{G} = \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \ldots \mathfrak{F}_r \supset \mathfrak{F}_{r+1} = \{1\},$$

then $\mathfrak{G}/\mathfrak{F}_{i+1}$ is an N-group of rank i, and $\mathfrak{F}_{i+1}$ is of rank $r - i$,

$$(i = 1, 2, \ldots, r - 1).$$

Let $\mathfrak{G}$ be an N-group with $\mathfrak{F}$-series

$$\mathfrak{G} = \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \ldots \supset \mathfrak{F}_{r+1} = \{1\},$$

and let F_i be a representative in $\mathfrak{G}$ of a generating element of $\mathfrak{F}_i$ modulo $\mathfrak{F}_{i+1}$: then any element G of $\mathfrak{G}$ may be written uniquely in the form

(2.5.1) $$G = F_1^{\alpha_1} F_2^{\alpha_2} \ldots F_r^{\alpha_r},$$

where $\alpha_1, \ldots, \alpha_r$ are integers, positive, negative or zero. In what follows we assume that an $\mathfrak{F}$-series, and the elements $F_1, \ldots, F_r$ have been selected once and for all. We refer to the elements $F_1, \ldots, F_r$ as an $\mathfrak{F}$-*basis* for $\mathfrak{G}$, and to the representation (2.5.1) as the $\mathfrak{F}$-*representation* of $\mathfrak{G}$. Because of (2.2(2)) we have, since $(\mathfrak{F}_i, \mathfrak{F}_j) \subseteq \mathfrak{F}_k$, where $k > \max(i, j)$,

$$F_i^{\alpha_i} F_j^{\alpha_j} = F_j^{\alpha_j} F_i^{\alpha_i} F_k^{\gamma_k} F_{k+1}^{\gamma_{k+1}} \ldots F_r^{\gamma_r} \tag{2.5.2}$$

for some $k > \max(i, j)$ $(i, j = 1, 2, \ldots r)$. In particular we observe that F_r belongs to the centre of $\mathfrak{G}$, and in particular

$$F_r^{\alpha_r} F_j^{\alpha_j} = F_j^{\alpha_j} F_r^{\alpha_r} \tag{2.5.3}$$

for all j.

It is known that an N-group may be ordered, and indeed this follows easily from the existence of the $\mathfrak{F}$-representation (2.5.1) and the relations (2.5.2). We need only remark that if

$$G = F_1^{\alpha_1} \ldots F_r^{\alpha_r}, \quad H = F_1^{\beta_1} \ldots F_r^{\beta_r},$$

we may define $G < H$ when, for $1 \leqslant s \leqslant r$, $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots \alpha_{s-1} = \beta_{s-1}$, but $\alpha_s < \beta_s$. With this lexicographic ordering, it follows from (2.5.2) that if $G < H$ then $GK < HK$, $KG < KH$, for all $K \in \mathfrak{G}$, from which it follows that

$$\text{if } G_1 < H_1 \text{ and } G_2 < H_2, \text{ then } G_1G_2 < H_1H_2, \tag{2.5.4}$$

which is the condition that the relation "$<$" order the group.

3. The group ring of an N-group. For a moment, let $\mathfrak{G}$ be any group and let Φ be any field of characteristic 0. The (discrete) group ring Γ of $\mathfrak{G}$ over Φ will consist of all finite sums of the form

$$x = \sum \xi_i G_i, \qquad \xi_i \in \Phi,\ G_i \in \mathfrak{G}, \tag{3.0.1}$$

with addition, scalar multiplication, and ring multiplication defined in the natural manner. Γ may of course be of infinite rank over Φ, but in considerations involving only a finite number of elements of Γ of the form (3.0.1), we may assume that the summations run over the same group elements in each case. As usual we identify the prime subfield of Φ with the rationals, and the unit elements of Γ and of $\mathfrak{G}$ with the number 1. Thus there will be no confusion in supposing that, in an expression such as (2.5), the rational integers α_i belong to Φ: indeed we will often write, for example, if $F \in \mathfrak{G}$,

$$F^{\alpha} = 1 + \alpha(F - 1) + \tfrac{1}{2}\alpha(\alpha - 1)(F - 1)^2 + \ldots,$$

where α is a positive integer and F^{α} is considered as an element of Γ.

Let $\mathfrak{G}_1$ be any subgroup of $\mathfrak{G}$: then the group ring Γ_1 of $\mathfrak{G}_1$ over Φ may be considered as a subalgebra of Γ. If $\mathfrak{G}_1$ is normal in $\mathfrak{G}$, and if $\mathfrak{G}' = \mathfrak{G}/\mathfrak{G}_1$, we may consider also the group ring Γ' of $\mathfrak{G}'$ over Φ. We recall (**6**, §4) first the relationship between Γ, Γ_1, Γ'.

Let Δ be the two-sided ideal of Γ spanned by all elements of the form $(G - 1)$, $G \in \mathfrak{G}$. A necessary and sufficient condition that an element $x \in \Gamma$,

$$x = \sum \xi_i G_i,$$

belong to Δ is that $\sum \xi_i = 0$, and clearly $\Gamma/\Delta \cong \Phi$ so that Δ is a maximal ideal of Γ. Similarly Δ_1 is the maximal ideal of Γ_1 spanned by the elements $G_1 - 1$ for all $G_1 \in \mathfrak{G}_1$. The homomorphism of $\mathfrak{G}$ onto $\mathfrak{G}'$ defined by

$$G \rightarrow G\,\mathfrak{G}_1$$

may be extended in a natural fashion to a homomorphism of Γ onto Γ' by means of the mapping

$$x = \sum \xi_i G_i \rightarrow x' = \sum \xi_i G_i \mathfrak{G}_1. \tag{3.0.2}$$

The kernel of this homomorphism (3.0.2) may be identified as follows. Let $\bar{G}_\alpha$ be a representative in $\mathfrak{G}$ of the coset $G_\alpha \mathfrak{G}_1$ and let $x = \sum \xi_i G_i$ be written in the form

$$x = \sum_{\alpha,\beta} \xi'_{\alpha\beta} \bar{G}_\alpha G_{1\beta}, \qquad G_{1\beta} \in \mathfrak{G}_1$$

then the mapping (3.0.2) may be written

$$x = \sum_{\alpha,\beta} \xi'_{\alpha\beta} \bar{G}_\alpha G_{1\beta} \rightarrow \sum_\alpha \left(\sum_\beta \xi'_{\alpha\beta} \right) G_\alpha \mathfrak{G}_1$$

and $x \rightarrow 0$ if and only if

$$\sum_\beta \xi'_{\alpha\beta} = 0$$

for all α. Hence if $x \rightarrow 0$, we may write

$$x = \sum_{\alpha,\beta} \xi'_{\alpha\beta} G_\alpha (G_{1\beta} - 1), \tag{3.0.3}$$

and conversely. Thus, we have proved

LEMMA 3.1. *The kernel of the homomorphism of* Γ *upon* Γ' *defined by* (3.0.2) *is the ideal* $\Gamma\Delta_1$, *where* Δ_1 *is the ideal of* Γ_1 *spanned by all elements of the form* $(G_{1\beta} - 1)$, $G_{1\beta} \in \mathfrak{G}_1$.

Suppose next that $\mathfrak{G}$ is an *ordered* group satisfying (2.5.4). Then Γ contains no divisors of zero, and no units other than scalar multiples of group elements. For consider two elements $x, y \neq 0$ of Γ;

$$x = \sum_{i=1}^{m} \alpha_i G_i, \qquad \alpha_1 \neq 0,\ \alpha_m \neq 0,$$

$$y = \sum_{j=1}^{n} \beta_j H_j, \qquad \beta_1 \neq 0,\ \beta_n \neq 0.$$

We may suppose $G_1 < G_2 < \ldots < G_m$, $H_1 < H_2 < \ldots < H_n$. Then in the product

$$xy = \alpha_1\beta_1 G_1 H_1 + \ldots + \alpha_m \beta_n G_m H_n$$

we have $G_1H_1 < G_iH_j < G_mH_n$ $(1 < i < m,\ 1 < j < m)$, so that if $xy = 0$, $\alpha_1\beta_1 = \alpha_m\beta_n = 0$, which is false, and hence $xy \neq 0$ for all m, n. Similarly if $xy = 1$, then $m = n = 1$ and $\alpha_1\beta_1 = 1$, $G_1H_1 = 1$, $H_1 = G_1^{-1}$.

In particular, since any N-group is ordered, we may state:

THEOREM 3.2. *The group ring Γ of an N-group $\mathfrak{G}$ has no proper divisors of zero, and no units other than scalar multiples of group elements.*

4. The structure of the ideal Δ. From now on $\mathfrak{G}$ will be an N-group. Let Δ be the ideal spanned by the elements $G - 1$, for $G \in \mathfrak{G}$ as in §3. Consider the element $G - 1 \in \Delta$. Because of (2.5.1), we may write

$$(G - 1) = (F_1^{\alpha_1} F_2^{\alpha_2} \dots F_r^{\alpha_r} - 1), \tag{4.0.1}$$

and because of the identity

$$AB - 1 = (A - 1) + (B - 1) + (A - 1)(B - 1), \tag{4.0.2}$$

we may write $G - 1$ as a linear combination,

$$G - 1 = \sum_{\omega, \rho} \delta_{\omega,\rho}\pi_{\omega,\rho}. \tag{4.0.3}$$

Here the coefficients $\delta_{\omega,\rho}$ are integers and

$$\pi_{\omega,\rho} = (F_{i_1}^{\alpha_{i_1}} - 1)(F_{i_2}^{\alpha_{i_2}} - 1)\dots(F_{i_s}^{\alpha_{i_s}} - 1), \tag{4.0.4}$$

$(i_1, \dots i_s)$ is a subset of the integers $1, 2, \dots, r$ with

$$1 \leq i_1 < i_2 < \dots i_s \leq r.$$

Also $\alpha_{i_1}, \dots, \alpha_{i_s}$ are the exponents of $F_{i_1}, \dots, F_{i_s}$ in (4.0.1), where for brevity we have written $\rho = (i_1, \dots, i_s)$ and

$$\omega = (\alpha_{i_1}, \dots, \alpha_{i_s}).$$

Note also that the summation in (4.0.3) will extend over certain subsets ρ and ω determined by the $(\alpha_1, \dots, \alpha_r)$ of (4.0.1) and the identity (4.0.2).

We have also the binomial identity, for integers α:

$$\begin{aligned}(A^\alpha - 1) &= \{1 + (A^{\pm 1} - 1)\}^{|\alpha|} - 1 \\ &= \sum_t \binom{|\alpha|}{t}(A^{\pm 1} - 1)^t, \qquad t = 1, 2, \dots |\alpha|,\end{aligned} \tag{4.0.5}$$

the positive or negative exponent on the right being taken as α is positive or negative.

Consider now a product of the form

$$(F_{i_1}^{\pm 1} - 1)^{\delta_1}(F_{i_2}^{\pm 1} - 1)^{\delta_2}\dots(F_{i_s}^{\pm 1} - 1)^{\delta_s} \tag{4.0.6}$$

where $1 \leq i_1 < i_2 < \dots < i_s \leq r$ as in (4.0.4), $\delta_1, \dots, \delta_s$ are positive integers, and where any combination of positive and negative exponents may occur, except that of course two factors

$$(F_j - 1) \text{ and } (F_j^{-1} - 1)$$

do not occur in the same product. We call $w = \delta_1 + \ldots + \delta_s$ the *degree* of such a product, and denote by

$$P_{w1}, P_{w2}, \ldots, P_{wl_w}, \qquad w = 1, 2, \ldots,$$

the various formally distinct products of degree w, since for fixed w there are only a finite number of such distinct products (4.0.6).

Applying (4.0.5) to each of the factors

$$(F_{i_k}^{\alpha_{i_k}} - 1)$$

in (4.0.4), we readily verify that every $\pi_{\omega,\rho}$, and hence by (4.0.3) every element $(G - 1)$, may be written as a linear combination, with integral coefficients, of products (4.0.6). Now every element x of Δ is expressible in the form

$$x = \sum \xi_i (G_i - 1), \qquad \xi_i \in \Phi,$$

and we see therefore that any element x of Δ may be written as a linear combination of products of the form (4.0.6),

$$(4.0.7) \qquad x = \gamma_1 P_{w_1 \nu_1} + \gamma_2 P_{w_2 \nu_2} + \ldots + \gamma_s P_{w_s \nu_s}, \qquad \gamma_i \in \Phi.$$

It is readily verified that the products $P_{w\nu}$ are linearly independent: for suppose

$$\gamma_1 P_{w_1 \nu_1} + \gamma_2 P_{w_2 \nu_2} + \ldots + \gamma_k P_{w_k \nu_k} = 0, \qquad \gamma_i \in \Phi,$$

where the products are distinct and all $\psi_i \neq 0$. An easy lexicographical argument similar to that used in establishing (3.2) shows that a relation of this type would imply one of the same kind among group elements of the form

$$G_i = F_1^{\alpha_{1i}} \ldots F_r^{\alpha_{ri}}$$

where the $(\alpha_{1i}, \ldots, \alpha_{ri})$ are all distinct, which is impossible in view of the uniqueness of the representation (2.5.1). We omit the details, which may be supplied without difficulty. Thus we have proved

THEOREM 4.1. *The formally distinct products $P_{w\nu}$ of the form* (4.0.6) *are linearly independent and form a basis for the ideal Δ. The representation* (4.0.7) *is unique.*

As an immediate consequence of (4.1) we have:

COROLLARY 4.2. *A relation of the form*

$$p_0 + (F_\rho - 1)p_1 + \ldots + (F_\rho - 1)^n p_n = 0$$

where $p_0, p_1, \ldots, p_n$ are elements of the group ring of $\mathfrak{F}_{\rho+1}$ implies $p_k = 0$ $(k = 0, 1, \ldots, n)$.

We define the *weight* of any product of w factors $(F_{\rho_i}^{\pm 1} - 1)$:

$$(4.2.0) \qquad \prod (F_{\rho_i}^{\pm 1} - 1), \qquad i = 1, 2, \ldots, w,$$

where the factors occur in any order, and where

$$(F_{\rho_i} - 1), \quad (F_{\rho_i}^{-1} - 1)$$

may occur in the same product, to be

$$W = \sum 2^{\rho_i}, \tag{4.2.1}$$

where the summation is taken over the same ρ_i, $(i = 1, 2, \ldots, w)$ which occur in (4.2.0).

In particular, therefore, the weight W of a product

$$(F_1^{\pm 1} - 1)^{\alpha_1}(F_2^{\pm 1} - 1)^{\alpha_2} \ldots (F_r^{\pm 1} - 1)^{\alpha_r}$$

(α_i non-negative integers) will be

$$W = \alpha_1 2^1 + \alpha_2 2^2 + \ldots + \alpha_r 2^r.$$

Consider now the identity

$$\begin{aligned}(B-1)(A-1) &= (A-1)(B-1) + AB(B^{-1}A^{-1}BA - 1)\\ &= (A-1)(B-1) + (B^{-1}A^{-1}BA - 1)\\ &\quad + (A-1)(B^{-1}A^{-1}AB - 1)\\ &\quad + (B-1)(B^{-1}A^{-1}BA - 1)\\ &\quad + (A-1)(B-1)(B^{-1}A^{-1}BA - 1).\end{aligned} \tag{4.2.2}$$

If $\rho_i < \rho_j$, we have, because of (2.5.2),

$$F_{\rho_j}^{\mp 1} F_{\rho_i}^{\mp 1} F_{\rho_j}^{\pm 1} F_{\rho_i}^{\pm 1} = \prod_\sigma F_\sigma^{a_\sigma}, \qquad \sigma = \rho_j + 1,\ \rho_j + 2, \ldots, r;$$

and hence by (4.0.7)

$$(F_{\rho_j}^{\mp 1} F_{\rho_i}^{\mp 1} F_{\rho_j}^{\pm 1} F_{\rho_i}^{\pm 1} - 1) = \sum \gamma_{wv} P^*_{wv}, \qquad \gamma_{wv} \in \Phi,$$

where P^*_{wv} are products of the form (4.0.6) with all $\rho > \rho_j$. Now the weight $W(P^*_{wv})$ of every product P^*_{wv} is at least 2^{ρ_j+1}, and since

$$W((F_{\rho_j}^{\pm 1} - 1)(F_{\rho_i}^{\pm 1} - 1)) = 2^{\rho_i} + 2^{\rho_j} < 2^{\rho_j+1},$$

we see, by using (4.2.2) with

$$A = F_{\rho_i}^{\pm 1},\ B = F_{\rho_j}^{\pm 1},$$

that

$$(F_{\rho_j}^{\pm 1} - 1)(F_{\rho_i}^{\pm 1} - 1) = (F_{\rho_i}^{\pm 1} - 1)(F_{\rho_j}^{\pm 1} - 1) + \text{products } P_{wv},$$

where

$$W(P_{wv}) > 2^{\rho_i} + 2^{\rho_j}.$$

By repeated application of this "straightening" process it may be verified that *every product of the form* (4.2.0) *of weight W may be expressed as a linear combination of products of weights no less than W in which the factors occur in the natural order.*

We observe that in these "straightened" products we may have consecutive factors of the form

$$(F_\rho^{-1} - 1)^{\alpha_\rho}(F_\rho - 1)^{\beta_\rho}.$$

For convenience let us denote a "straightened" product of weight W (containing perhaps both

$$(F_\rho^{-1} - 1)^{\alpha_\rho}, \quad (F_\rho - 1)^{\beta_\rho}$$

with $\alpha_\rho, \beta_\rho \neq 0$) by

$$Q_W = \prod_\rho (F_\rho^{-1} - 1)^{\alpha_\rho}(F_\rho - 1)^{\beta_\rho}, \qquad \rho = 1, 2, \ldots, r, \tag{4.2.3}$$

where

$$W = \sum_\rho (\alpha_\rho + \beta_\rho)\, 2^\rho,$$

and some of α_ρ, β_ρ may be zero.

We establish now the principal theorem of this section:

THEOREM 4.3. *For all w, Δ^{w+1} is a proper ideal of Δ^w, and $\bigcap \Delta^w = 0$, i.e.*

$$\Delta \supset \Delta^2 \supset \ldots \supset \Delta^w \supset \Delta^{w+1} \supset \ldots$$

and the only element in Δ^w for all w is 0.

Proof. Since $(F_1 - 1)^w \neq 0 \in \Delta^w$ for all w, it is clear that no power of Δ vanishes, and it will be enough to prove the following: "the only element x which belongs to Δ^w for all w is zero." The proof is by induction over the rank of $\mathfrak{G}$, and we show first that the theorem is true for groups of rank 1, that is, when $\mathfrak{G} = \{F\}$.

Suppose that $x \neq 0$ is an element of $\Delta(\mathfrak{G})$, where $\mathfrak{G} = \{F\}$, which is in Δ^u for arbitrarily large u. Now by (4.1) we may write x uniquely in the form

$$x = \sum \gamma_i(F^{-1} - 1)^i + \sum \delta_j(F - 1)^j, \qquad i = 1, \ldots, n;\ j = 1, \ldots, m.$$

If $x \in \Delta^u$ for arbitrary u, $y = F^\rho x$ has the same property for any integer ρ, and hence by choosing ρ large enough to cancel all negative powers in x above we may suppose that if there is an element $y \neq 0$ in Δ^u for all u it has the form

$$y = \gamma_1(F - 1)^{\alpha_1} + \ldots + \gamma_n(F - 1)^{\alpha_n}, \qquad \gamma_i \in \Phi,$$

where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$ and $\gamma_k \neq 0$ $(k = 1, \ldots, n)$. Choose $u > \alpha_1$: then if $y \in \Delta^u$, y may be written as a finite sum of products with $\geqslant u$ factors:

$$y = \sum_i \delta_i\, (F^{-1} - 1)^{\beta_i}(F - 1)^{\beta'_i}, \qquad \beta_i + \beta'_i \geqslant u > \alpha_1,$$

and hence

$$\gamma_1(F - 1)^{\alpha_1} + \ldots + \gamma_n(F - 1)^{\alpha_n} - \sum \delta_i\, (F^{-1} - 1)^{\beta_i}(F - 1)^{\beta'_i} = 0.$$

Now

$$(F^{-1} - 1)^{\beta_i} = \pm F^{-\beta_i}(F - 1)^{\beta_i},$$

and hence

$$(F-1)^{\alpha_1}[\gamma_1+\gamma_2(F-1)^{\alpha_2-\alpha_1}+\ldots+\gamma_n(F-1)^{\alpha_n-\alpha_1}$$
$$-\sum(\pm\delta_i)\,F^{-\beta_i}(F-1)^{\beta_i+\beta'_i-\alpha_1}]=0.$$

Since there are no divisors of zero in Γ, we have

$$[\gamma_1+\gamma_2(F-1)^{\alpha_2-\alpha_1}+\ldots\sum\pm\delta_i(F-1)^{\beta_i+\beta'_i-\alpha_1}F_i^{-\beta_i}]=0,$$

which implies $\gamma_1 \equiv 0 \pmod{\Delta}$, since $\beta_i + \beta'_i - \alpha_1 > 0$, and hence $\gamma_i = 0$, which is contra hypothesis.

We now assume our theorem for groups of rank $\leqslant r-1$, (and in particular for $\mathfrak{F}_2$), and prove it for groups $\mathfrak{G}$ of rank r. Suppose that $x \neq 0$ is an element of $\Delta(\mathfrak{G})$ which is in $\Delta^u(\mathfrak{G})$ for arbitrary large u. By (4.1) we have a unique expression

$$x=\sum\gamma_{uk}P_{uk}, \qquad \gamma_{uk}\in\Phi,$$

which we write

$$x=p_0+(F_1^{\pm1}-1)^{\alpha_1}p_1+\ldots+(F_1^{\pm1}-1)^{\alpha_n}p_n,$$

where p_0 is a sum of P_{uk} lying in $\Delta(\mathfrak{F}_2)$ of the form

$$P_{uk}=\prod(F_\rho^{\pm1}-1)^{\alpha_\rho}, \qquad \rho\geqslant 2, \tag{4.3.1}$$

and p_i, $i \geqslant 1$, is a similar sum with perhaps a term containing only an element of Φ. If x belongs to $\Delta^u(\mathfrak{G})$ for all u, so does $y = F^e \,.\, x$, and we may once more suppose, therefore, that our x has the form

$$x=q_0+(F_1-1)^{\alpha_1}q_1+\ldots+(F_1-1)^{\alpha_n}q_n,$$

where $0<\alpha_1<\alpha_2<\ldots<\alpha_n$, and q_k $(k=0,\ldots,n)$ is also a sum of P_{uk} of the form (4.3.1) with $q_1, q_2, \ldots, q_n \neq 0$ and where q_0 may or may not be zero for the moment. Consider $q_0, \ldots, q_n$: they are all in $\Delta(\mathfrak{F}_2)$ since otherwise (4.3) is false for $\mathfrak{G}/\mathfrak{F}_2$. By our induction assumption if $q_0 \neq 0$ there is an u' such that $q_0 \notin \Delta^{u'}(\mathfrak{F}_2)$; if $q_0 = 0$ there is, since $q_1 \neq 0$, a u'' such that $q_1 \notin \Delta^{u''}(\mathfrak{F}_2)$. Let $w = u'$ or $u'' + \alpha_1$ as $q_0 \neq 0$, or $q_0 = 0$, and choose $u = 2^r w$.

If $x \in \Delta^u(\mathfrak{G})$, x may be written as a sum of products of the form (4.2.0) $p'(u_i)$, each having at least u factors and, therefore, of weight $u_i \geqslant u$:

$$x=\sum\gamma(u_i)\prod_{j=1}^{u_i}(F_{\rho_j}^{\pm1}-1)=\sum\gamma(u_i)\,p'(u_i), \qquad \gamma(u_i)\in\Phi,$$

where $u_i \geqslant u$. Now as in (4.2.3) each of the products $p'(u_i)$ in this sum may be "straightened" without loss of weight so that

$$x=\sum\bar{\gamma}(v_i)\,Q(v_i), \qquad \bar{\gamma}(v_i)\in\Phi,$$

where the weights v_i of all $Q(v_i)$ are at least u. These products $Q(v_i)$ are of two types: those lying in $\Delta(\mathfrak{F}_2)$ and those containing a factor

$$(F_1^{-1}-1)^{\beta_i}\,(F_1-1)^{\beta'_i}$$

at the beginning. Denote a $Q(v_i)$ in $\Delta(\mathfrak{F}_2)$ by $Q'(v_i)$: then each $Q(v_i)$ is either a $Q'(v_i)$ or of the form

$$(F_1^{-1} - 1)^{\beta_i} \cdot (F_1 - 1)^{\beta'_i} \cdot Q'(v_j).$$

Because of our choice of u, every product $Q'(u_i)$ of weight $\geq u$ contains at least w factors, since the weight in $\Delta(\mathfrak{F}_2)$ of a product $Q'(u_i)$ is of the form (cf. (4.2.3)):

$$(\alpha_2 + \beta_2)2^1 + (\alpha_3 + \beta_3)2^2 + \ldots + (\alpha_r + \beta_r)2^{r-1},$$

and if this is greater than or equal to $u = 2^r w$, then certainly

$$(\alpha_2 + \beta_2) + \ldots + (\alpha_r + \beta_r) \geq w.$$

Hence each $Q'(u_i)$ is in $\Delta^w(\mathfrak{F}_2)$. Now we have

$$x = q_0 + (F_1 - 1)^{\alpha_1} q_1 + \ldots = \sum \{\gamma'_k Q'(u_k) + \delta_k (F_1^{-1} - 1)^{\beta_k} (F_1 - 1)^{\beta'_k} Q'(v_k)\},$$

where the right-hand side is in $\Delta^u(\mathfrak{G})$. If $q_0 \neq 0$ then

$$q_0 = \sum \gamma'_k Q'(u_k)$$

by (4.2). This is impossible, since $Q'(u_k) \in \Delta^w(\mathfrak{F}_2)$, and $w = u'$, where $q_0 \notin \Delta^{u'}(\mathfrak{F}_2)$. If $q_0 = 0$ we have

$$\sum \gamma'_k Q'(u_k) = 0$$

and hence,

$$(F_1 - 1)^{\alpha_1}[q_1 + (F_1 - 1)^{\alpha_2 - \alpha_1} q_2 + \ldots - (\sum \bar{\delta}_k Q'(\bar{u}_k) + \ldots)] = 0,$$

for it follows that $\beta_k + \beta_k' \geq \alpha_1$ for all k since if, for some k, $\beta_k + \beta_k' < \alpha_1$ we could remove the factor

$$(F_1 - 1)^{\beta_k + \beta'_k}$$

and get $Q'(u_k) = 0$. Again this implies a relation

$$q_1 = \sum \bar{\delta}_k Q'(\bar{u}_k)$$

where $Q'(\bar{u}_k)$ has at least $(u'' + \alpha_1) - \alpha_1$ factors and is therefore in $\Delta^{u''}(\mathfrak{F}_2)$ which is impossible, since $q_1 \notin \Delta^{u''}(\mathfrak{F}_2)$.

We have proved, therefore, that 0 is the only element in $\Delta^w(\mathfrak{G})$ for all w. It follows that for every element $x \neq 0 \in \Delta$ there exists a w such that $x \in \Delta^w$, $x \notin \Delta^{w+1}$. Moreover, Γ/Δ^{w+1} is an algebra of finite rank over Φ.

In view of (4.3) it is natural to introduce infinite sums into the ring Γ by means of a "Δ-adic" topology. Take the set $\{\Delta^w\}$ ($w = 1, 2, \ldots$) as a fundamental system of neighbourhoods of the element 0 in Γ: then a sequence $a_1, a_2, \ldots, a_n \ldots$ of elements of Γ "converges" to $a \in \Gamma$ if, for given w, there exists an integer N such that $n > N$ implies that

$$(a_n - a) \in \Delta^w.$$

Let Γ^* be the completion of Γ in this topology, and let Δ^* be the completion

of Δ. Clearly we may consider Γ^* to be the ring of all "power series" a^* of the form

$$a^* = \alpha_0 + \sum \alpha_k d_k, \qquad \alpha_k \in \Phi, k = 1, 2, \ldots, \tag{4.3.2}$$

where $d_k \in \Delta^k$, while Δ^* consists of all elements a^* with $\alpha_0 = 0$. As usual, we identify Γ with its isomorphic image in Γ^*. We note that

$$G^{-1} = 1 - (G-1) + (G-1)^2 - (G-1)^3 + \ldots, \tag{4.3.3}$$

in Γ^*, and that, more generally, if $a^* \notin \Delta^*$ then a^* is a unit in Γ^*.

It is clear that we have the following:

THEOREM 4.4. *For any integer n, $\Gamma/\Delta^{n+1} \cong \Gamma^*/\Delta^{*(n+1)}$ and $\bigcap \Delta^{*w} = 0$.*

Because of (4.4) we can work with either Γ^* or Γ in what follows.

5. The dimensional sub-groups of $\mathfrak{G}$. The concept of dimensional subgroup relative to the ideal Δ (or Δ^*) is now natural. Let $\mathfrak{D}_w$ ($w = 1, 2, \ldots$) be the set of all elements $D_w \in \mathfrak{G}$ such that

$$D_w \equiv 1 \pmod{\Delta^{*w}}.$$

If $D_w \equiv D_w' \equiv 1 \pmod{\Delta^{*w}}$ we have

$$D_w = 1 + d_w,\ D_w' = 1 + d_w'; \qquad d_w, d_w' \in \Delta^{*w}$$

and hence

$$D_w D_w' = 1 + d_w + d_w' + d_w d_w' \equiv 1 \pmod{\Delta^{*w}},$$

while

$$D^{-1} = 1 - d_w + d_w^2 - d_w^3 + \ldots \equiv 1 \pmod{\Delta^{*w}}.$$

Further, if G is any element of $\mathfrak{G}$, we have

$$G^{-1}D_wG = 1 + G^{-1}d_wG \equiv 1 \pmod{\Delta^{*w}},$$

so that $\mathfrak{D}_w$ is a normal subgroup of $\mathfrak{G}$. We have also

$$D_w^{-1}D_v^{-1}D_wD_v = 1 + D_w^{-1}D_v^{-1}(d_wd_v - d_vd_w) \equiv 1 \pmod{\Delta^{*w+v}},$$

where $D_v = 1 + d_v$, $d_v \in \Delta^{*v}$, and hence $(\mathfrak{D}_w, \mathfrak{D}_v) \subseteq \mathfrak{D}_{w+v}$. In particular $(\mathfrak{D}_w, \mathfrak{G}) \subseteq \mathfrak{D}_{w+1}$. These results may be summarized in

THEOREM 5.1. *The set $\mathfrak{D}_w$ of elements D_w of $\mathfrak{G}$ which are congruent to* 1 mod Δ^{*w} *form a normal subgroup of $\mathfrak{G}$, and the series*

$$\mathfrak{G} = \mathfrak{D}_1 \supseteq \mathfrak{D}_2 \supseteq \ldots$$

is a central series of $\mathfrak{G}$ with the property $(\mathfrak{D}_w, \mathfrak{D}_v) \subseteq \mathfrak{D}_{w+v}$ ($w, v = 1, 2, \ldots$).

The subgroups $\mathfrak{D}_w$ we shall call **(6; 7; 8)** the *dimensional subgroups of $\mathfrak{G}$.*

Suppose D_w belongs to $\mathfrak{D}_w$ but not to $\mathfrak{D}_{w+1}$. We have

$$D_w = 1 + d_w$$

where $d_w \in \Delta^{*w}$ and $d_w \notin \Delta^{*w+1}$, and hence

$$D_w^n = (1 + d_w)^n \equiv 1 + nd_w \bmod \Delta^{*w+1}.$$

Now since Φ is of characteristic 0, $nd_w \not\equiv 0$ (Δ^{*w+1}) for any integer n, and hence

$$D_w^n \in \mathfrak{D}_{w+1}$$

only if $n = 0$. Hence if $\mathfrak{D}_w \neq \mathfrak{D}_{w+1}$, $\mathfrak{D}_w/\mathfrak{D}_{w+1}$ is torsion free. By refining the dimensional series (5.1) we obtain an $\mathfrak{F}$-series of the form (2.2.0). It follows that the dimensional series of $\mathfrak{G}$ is of finite length. We have, therefore:

THEOREM 5.2. *The dimensional subgroups of $\mathfrak{G}$ form a central series of finite length $\mathfrak{G} = \mathfrak{D}_1 \supseteqq \mathfrak{D}_2 \supseteqq \ldots \supseteqq \mathfrak{D}_m \supseteqq \mathfrak{D}_{m+1} = \{1\}$; and are such that if $\mathfrak{D}_w \supset \mathfrak{D}_{w+1}$ then $\mathfrak{D}_w/\mathfrak{D}_{w+1}$ is a direct product of infinite cyclic groups.*

The problem of identifying the dimensional subgroups of $\mathfrak{G}$ we leave to a later paper. However, in view of (**6**, Theorem 5.5), it is natural to suspect that they are the minimal subgroups enjoying the properties indicated in (5.2), and this is indeed the case, as we shall prove.

PART II: THE LIE ALGEBRA OF AN N-GROUP

6. The Campbell-Hausdorff Formula in Γ^*. For the rest of this paper we work with Γ^* and Δ^*, since the possibility of infinite sums, which was inconvenient in most of Part I, is now essential.

Let x^* be any element of Δ^*: then we may form

$$\exp(x^*) = 1 + x^* + \frac{x^{*2}}{2!} + \ldots + \frac{x^{*n}}{n!} + \ldots \tag{6.0.1}$$

where the series on the right certainly converges in the Δ-adic topology of §4 to an element of Γ^*. Clearly $X^* = \exp x^*$ is a unit in Γ^*, and

$$\exp(\alpha x^*) \exp(\beta x^*) = \exp(\alpha + \beta)x^* \tag{6.0.2}$$

for all $\alpha, \beta \in \Phi$. We note that if $\exp x^* = 1$, then $x^* = 0$, for $(\exp x^*) - 1 = x^* u = 0$, where

$$u = 1 + \frac{x^*}{2!} + \frac{x^{*2}}{3!} + \ldots$$

is a unit in Γ^*.

Similarly, if $y^* \in \Delta^*$ we may define

$$\log(1 + y^*) = y^* - \tfrac{1}{2}y^{*2} + \tfrac{1}{3}y^{*3} - \tfrac{1}{4}y^{*4} + \ldots, \tag{6.0.3}$$

where again the series on the right converges to an element of Δ^*. It is clear that $\exp \log (1 + y^*) = 1 + y^*$ and $\log \exp x^* = x^*$. In general, of course, exponentials and logarithms will not be defined for all elements of Γ^*, since for example the existence of $\exp (\alpha + x^*)$ would imply the existence of

$\exp \alpha$ in Φ. In particular, however, $\log G$ is defined for all elements $G \in \mathfrak{G}$, and we have

(6.0.4) $$g = \log G = (G - 1) - \tfrac{1}{2}(G - 1)^2 + \ldots, \quad G = \exp g.$$

Now with Δ^* we may associate the Lie algebra $\Lambda^* = (\Delta^*)_l$ in the usual fashion by defining the binary operation of commutation in $(\Delta^*)_l$ by means of

$$x^* \circ y^* = x^*y^* - y^*x^*,$$

for all $x^*, y^* \in \Delta^*$. Clearly Λ^* is of infinite rank over Φ. Let Θ^* be an ideal of Δ^*, and let $M^* = (\Theta^*)_l$; then M^* is an ideal of the Lie algebra Λ^*. In particular we have

(6.0.5) $$\Lambda^* = (\Delta^*)_l \supset (\Delta^{*2})_l \supset (\Delta^{*3})_l \supset \ldots.$$

Let us define the "lower central series" of Λ^* by setting $\Lambda^* = \Lambda^*_1$, and $\Lambda^*_{k+1} = \Lambda^*_k \circ \Lambda^*$, where $\Lambda^*_k \circ \Lambda^*$ is the ideal spanned by all elements of the form $l^*_k \circ l^*$ with $l^*_k \in \Lambda^*_k$ and $l^* \in \Lambda^*$. Then certainly

$$\Lambda^*_w \subseteq (\Delta^{*w})_l, \qquad w = 1, 2, \ldots,$$

so that

(6.0.6) $$\Lambda^* = \Lambda^*_1 \supset \Lambda^*_2 \supset \ldots \supset \Lambda^*_w \supset \Lambda^*_{w+1} \supset \ldots,$$

and the only element belonging to Λ^*_w for all w is zero. Hence Λ^* is "generalized nilpotent" in the usual sense. We note also

LEMMA 6.1. *If $l^*_k \in \Lambda^*_k$ $(k = 1, 2, \ldots)$, then the infinite series*

$$l^*_1 + l^*_2 + \ldots + l^*_k + \ldots$$

is an element of Λ^*.

Proof. Since $l^*_k \in \Lambda^*_k \subseteq (\Delta^{*k})_l$, l^*_k, considered as an element of Δ^*, belongs to Δ^{*k}, so that the series in (6.1) converges to an element in Δ^* and therefore is an element of $(\Delta^*)_l = \Lambda^*$.

Now as we have seen in §4, if $X = 1 + x^*$, $Y = 1 + y^*$, where $x^*, y^* \in \Delta^*$, then

$$XY = 1 + x^* + y^* + x^*y^* = 1 + z^*,$$
$$X^{-1} = 1 - x^* + x^{*2} - x^{*3} + \ldots = 1 + \bar{x}^*,$$

where $z^*, \bar{x}^*$ belong to Δ^*, so that the set of all elements of the form $1 + x^*$ forms a group $\mathfrak{G}^*$ under multiplication. Indeed, $\mathfrak{G}^*$ is a normal subgroup of the unit group of Γ^*, consisting of all elements of the form $\alpha + x^*$, where $\alpha \neq 0 \in \Phi$ and $x^* \in \Delta^*$. The group $\mathfrak{G}^*$ is generalized nilpotent, for if $\mathfrak{D}^*_k$ is the normal subgroup of $\mathfrak{G}^*$ consisting of elements of the form $1 + x^*_k$, where $x^*_k \in \Delta^{*k}$, then

(6.1.1) $$\mathfrak{G}^* = \mathfrak{D}^*_1 \supset \mathfrak{D}^*_2 \supset \mathfrak{D}^*_3 \supset \ldots$$

is a central series of $\mathfrak{G}^*$, and

(6.1.2) $$\mathfrak{D}^*_w \supseteq \mathfrak{G}^*_w, \qquad w = 1, 2, \ldots,$$

where $\mathfrak{G}^*_w$ is the wth term of the lower central series (1.0.1) of $\mathfrak{G}^*$. Clearly,

therefore, $\mathfrak{G}^*_w \neq \mathfrak{G}^*_{w+1}$ for all w, and the only element common to all the $\mathfrak{G}^*_w$ will be the unit element. In particular $\mathfrak{G}$ itself is a subgroup of $\mathfrak{G}^*$.

We recall now the well-known Campbell-Hausdorff formula (**2**; **9**; **10**) which reveals the intimate connection between the group $\mathfrak{G}^*$ and the Lie algebra Λ^*. Let $X = 1 + x^*$, $Y = 1 + y^*$ be elements of $\mathfrak{G}^*$: then by (6.0.1):

$$X = \exp x, \quad Y = \exp y, \quad XY = \exp z, \tag{6.1.3}$$

where $x = \log(1 + x^*)$, $y = \log(1 + y^*)$ and $z = \log(1 + x^* + y^* + x^*y^*)$. Then the Campbell-Hausdorff formula gives z explicitly in terms of x and y as follows (**2**, formula 26; **9**, formula **10**):

$$z = x + y + \tfrac{1}{2}(x \circ y) + \tfrac{1}{12}(x \circ y \circ y) + \tfrac{1}{12}(y \circ x \circ x) + \tfrac{1}{24}(y \circ x \circ x \circ y) + \ldots, \tag{6.1.4}$$

where $x \circ y = xy - yx$ as usual, where we have written, for example, $x \circ y \circ y = (x \circ y) \circ y$, $y \circ x \circ x \circ y = ((y \circ x) \circ x) \circ y$, etc., and where the right side of (6.1.4) is an infinite sum of the type considered in (6.1) and therefore convergent. All the coefficients on the right are known to be rational and therefore belong to any Φ of characteristic 0.

For later use we note that, if $X^{-1}Y^{-1}XY = (X, Y) = \exp c$, then

$$c = (x \circ y) + \tfrac{1}{2}(x \circ y \circ x) + \tfrac{1}{2}(x \circ y \circ y) + \ldots, \tag{6.1.5}$$

where the right side is again an infinite sum of commutators in x and y. For convenience we introduce the notation

$$x[i_1, i_2, \ldots, i_s] = x_{i_1} \circ x_{i_2} \circ \ldots \circ x_{i_s}.$$

From (6.1.5) it follows that, if $X_i = \exp(x_i)$ and if $(X_1, X_2, \ldots, X_n) = \exp c_n$, then

$$c_n = (x_1 \circ x_2 \circ \ldots \circ x_n) + \sum l_w, \qquad w = n + 1, n + 2, \ldots \tag{6.1.6}$$

where each l_w is a rational linear combination of simple commutators of the form $x[i_1, i_2, \ldots, i_w]$ with $w > n + 1$, and where $i_1, i_2, \ldots, i_w$ is a permutation of the integers $1, 2, \ldots, n$ with repetitions allowed, but with each of $1, 2, \ldots n$ occuring at least once.

7. The Lie algebra of an N-group. Consider now the module $\mathfrak{L}$ of Λ^* spanned by all elements of the form $g = \log G$, where G runs over the N-group $\mathfrak{G}$. The principal theorem of this section is to the effect that $\mathfrak{L}$ is a Lie algebra of finite rank r over Φ, where r is the dimension of $\mathfrak{G}$. As a first step towards this result we prove:

LEMMA 7.1. *If $w > c$, the class of $\mathfrak{G}$, and if $g_i = \log G_i$, $i = 1, 2, \ldots w$ where $G_1, G_2, \ldots, G_w \in G$, then*

$$g_1 \circ g_2 \circ \ldots \circ g_w = 0.$$

Proof. By (6.1.6) we know that, if $w > c$,

$$1 = (G_1, G_2, \ldots, G_w) = \exp(g_1 \circ g_2 \circ \ldots \circ g_w + \textstyle\sum l_u),$$
$$u = w + 1, w + 2, \ldots,$$

where l_u is a linear combination of commutators of the form $g[i_1, i_2, \ldots, i_u]$, so that

(7.1.1) $$g_1 \circ g_2 \circ \ldots \circ g_w = -\sum l_u, \qquad u = w+1, w+2, \ldots.$$

Now every element of the form $g[i_1, i_2, \ldots, i_u] \in \Delta^{*u}$, so that (7.1.1) implies that *every* commutator $g_1 \circ g_2 \circ \ldots \circ g_w \in \Delta^{*w+1}$. In particular, every $g[i_1, i_2, \ldots, i_u] \in \Delta^{*u+1}$, and again (7.1.1) implies that $g_1 \circ g_2 \circ \ldots \circ g_w \in \Delta^{*w+2}$. Continuing in this way we can show that

$$g_1 \circ g_2 \circ \ldots \circ g_w \in \Delta^{*w+N}$$

for all values of N, and therefore $g_1 \circ g_2 \circ \ldots \circ g_w = 0$, since zero is the only element common to all Δ^{*w+N}.

It follows from (7.1) that the Lie algebra *generated* by all $g = \log G$ is nilpotent of class c. We prove now

LEMMA 9.2. *Let $F_1, F_2, \ldots, F_r$ be any $\mathfrak{F}$-basis for $\mathfrak{G}$ as in* (2.2), *and let $f_i = \log F_i$ $(i, = 1, 2, \ldots, r)$. Then if $g = \log G$, where $G \in \mathfrak{G}$, there exist rational numbers $\gamma_1, \gamma_2, \ldots \gamma_r$ so that*

$$g = \gamma_1 f_1 + \gamma_2 f_2 + \ldots + \gamma_r f_r.$$

In other words, the module $\mathfrak{L}$ spanned by the elements $g = \log G$ for all $G \in \mathfrak{G}$ is of finite rank r over Φ.

Proof. By an easy application of the Campbell-Hausdorff formula (6.1.4) we may show that

(7.2.1) $$\begin{aligned} F_1^{\alpha_1} F_2^{\alpha_2} \ldots F_r^{\alpha_r} &= \exp(\alpha_1 f_1) \exp(\alpha_2 f_2) \ldots (\exp a_r f_r) \\ &= \exp(\alpha_1 f_1 + \alpha_2 f_2 + \ldots + a_r f_r + \sum \beta_\rho c_\rho) \qquad \rho = 1, 2, \ldots, s, \end{aligned}$$

where $\alpha_1, \ldots, \alpha_r$ are integers, $\beta_1, \beta_2, \ldots \beta_s$ are rational, and $c_1, c_2, \ldots, c_s$ are commutators in $f_1, \ldots, f_r$ of the form $f[i_1, i_2, \ldots, i_t]$ with $t < c$. (Note that because of (7.1) only a finite number of commutators c_ρ will occur in (7.2.1).) Since every G may be written

$$G = F_1^{\alpha_1} \ldots F_r^{\alpha_r};$$

it will be sufficient to prove that, for all integers t,

(7.2.2) $$f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_t} = \sum \gamma_j f_j, \qquad j = 1, 2, \ldots, r$$

where the $\gamma_1, \ldots, \gamma_r$ are rational.

As in (4.2.1) let us define the weight of f_i to be 2^i, and the weight of a commutator $f[i_1, i_2, \ldots, i_t]$ to be

$$W(t) = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_t}.$$

Now because of (7.1) all commutators $f[i_1, i_2, \ldots, i_t]$ with $t > c$ vanish, and since $i_1, i_2, \ldots, i_t$ are all at most r, any commutator in the f_i of weight sufficiently great vanishes (e.g., any commutator of weight greater than $r^{2^r} = W_0$; so that W_0 is an upper bound to the weight of non-vanishing commutators, although by no means the least). Hence (7.2.2) is satisfied trivially for all

commutators $f[i_1, i_2, \ldots, i_t]$ of weight $> W_0$. Assume therefore that (7.2.2) holds for all commutators of weight greater than $W(t)$. Now by (6.1.6) and (7.1) we have

$$(7.2.3)\quad (F_{i_1}, F_{i_2}, \ldots, F_{i_t}) = \exp(f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_t} + \sum \beta'_\rho c'_\rho),$$
$$\rho = 1, 2, \ldots, s',$$

where β_ρ' are rational, and each c_ρ' is a commutator of the form $f[j_1, j_2, \ldots j_u]$ of weight $W(u)$ greater than $W(t)$ since $i_1, \ldots, i_t$ are contained among $j_1, j_2, \ldots, j_u$. By our assumption, therefore, $\sum \beta_\rho' c_\rho'$ can be expressed as a rational linear combination of the $f_1, \ldots, f_r$, and we re-write (7.2.3) in the form

$$(7.2.4)\quad (F_{i_1}, F_{i_2}, \ldots, F_{i_t}) = \exp(f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_t} + \sum \beta_j f_j),$$
$$j = 1, 2, \ldots, r.$$

On the other hand, by (2.5.2) we have

$$(7.2.5)\quad (F_{i_1}, F_{i_2}, \ldots, F_{i_t}) = F_k^{\alpha_k} F_{k+1}^{\alpha_{k+1}} \ldots F_r^{\alpha_r},$$

where $k > t - 1 + m$, $m = \max(i_1, i_2, \ldots, i_t)$. Consider any f_s with $s > t - 1 + m$; the weight of f_s is 2^s and we have, since certainly $i_1 \neq i_2$,

$$(7.2.6)\quad W = 2^{i_1} + \ldots + 2^{i_t} < t\,2^m < 2^{t-1}2^m < 2^s.$$

Now (7.2.5) may be re-written, as in (7.2.1), in the form

$$(7.2.7)\quad (F_{i_1}, F_{i_2}, \ldots, F_{i_t}) = \exp(\alpha_k f_k + \ldots + \alpha_r f_r + \sum \beta_\rho'' c_\rho'')$$

where β_ρ'' are rational and each c_ρ'' is a commutator of the form $f[k_1, k_2, \ldots, k_r]$ with $k_1, \ldots, k_r > k$, so that the weight of each c_ρ'' is greater than W by (7.2.6). By our assumption, therefore, we may express each c_ρ'' rationally in terms of $f_1, \ldots, f_r$ and (7.2.7) may be re-written

$$(7.2.8)\quad (F_{i_1}, F_{i_2}, \ldots, F_{i_t}) = \exp(\sum \delta_j f_j), \qquad j = 1, 2, \ldots, r,$$

with rational δ_j. Equating the right-hand sides of (7.2.3) and (7.2.8), we see that (7.2.2) holds for any commutator of weight W if it holds for all commutators of weight $> W$, which proves that (7.2.2) is true in general.

Since in particular we have proved that

$$(7.2.9)\quad f_i \circ f_j = \sum \gamma_{ijk} f_k, \qquad i, j, k = 1, 2, \ldots, r,$$

where the γ_{ijk} are rational, we may now state our main result:

THEOREM 7.3. *The elements $g = \log G$ for all $G \in \mathfrak{G}$ span a Lie algebra $\mathfrak{L}$ of rank r over Φ, where r is the dimension of $\mathfrak{G}$. A basis for $\mathfrak{L}$ may be taken as the set $f_1 = \log F_1, \ldots, f_r = \log F_r$, where $F_1, F_2, \ldots, F_r$ is any $\mathfrak{F}$-basis of $\mathfrak{G}$. The structure constants of $\mathfrak{L}$ relative to a basis of this type are rational.*

8. The Lie group associated with $\mathfrak{G}$. Suppose now that Φ is the real field. We show that $\mathfrak{G}^*$, the group of all elements of the form $1 + x^*$ with $x^* \in \Delta^*$,

now contains as a subgroup a real simply connected nilpotent Lie group $\mathfrak{A}$ whose Lie algebra is the algebra $\mathfrak{L}$ determined by $\mathfrak{G}$ as in (7.3). The N-group $\mathfrak{G}$ is a subgroup of $\mathfrak{A}$, so that in particular we establish a theorem of Malcev (**10**, theorem 6).

For any G belonging to the N-group $\mathfrak{G}$ let us define $\mathfrak{G}^\xi$ for real ξ by the equation

$$G^\xi = \exp(\xi g), \tag{8.0.1}$$

and let $\mathfrak{A}$ be the set of all such elements G^ξ. Because of (7.2) we may write

$$G^\xi = \exp(\xi_1 f_1 + \ldots + \xi_r f_r), \tag{8.0.2}$$

where $\xi_1, \ldots, \xi_r$ are real. If $G^\eta = \exp(\eta_1 f_1 + \ldots + \eta_r f_r)$ then we have $G^\xi G^\eta = G^\zeta$, where

$$G^\zeta = \exp(\zeta_1 f_1 + \ldots + \zeta_r f_r)$$

and

$$\zeta_i = p_i(\xi_1, \ldots, \xi_r;\ \eta_1, \ldots, \eta_r),$$

p_i being a polynomial in the $\xi_1, \ldots, \xi_r; \eta_1, \ldots, \eta_r$ determined in the usual fashion by the constants γ_{ijk} of (7.2.9), so that if we take $(\xi_1, \ldots, \xi_2)$ as the coordinates of G^ξ it is clear that $\mathfrak{A}$ is a Lie group. Indeed $(\xi_1 \ldots \xi_r)$ are canonical, and certainly the Lie algebra of $\mathfrak{A}$ is $\mathfrak{L}$. We obviously have $\mathfrak{G}$ as a discrete subgroup of $\mathfrak{A}$. Hence we have:

THEOREM 8.1. *The set of all elements* $\exp(\xi g)$, *where* ξ *is real and* $g = \log G$, $G \in \mathfrak{G}$, *forms under multiplication a real simply connected Lie group* $\mathfrak{A}$ *whose Lie algebra is the rational Lie algebra* $\mathfrak{L}$ *determined by* $\mathfrak{G}$. *In particular, therefore, every N-group* $\mathfrak{G}$ *is a discrete subgroup of a Lie group* $\mathfrak{A}$ *with rational Lie algebra.*

REFERENCES

1. P. Hall, *A contribution to the theory of groups of prime power orders*, Proc. London Math. Soc. (2), *36* (1933), 29–95.
2. F. Hausdorff, *Die symbolische Exponentialformel in der Gruppentheorie*, Sitz. der Sachsischen Akad. Wiss. (Math.-phys. Klasse), *58* (1906), 19–48.
3. K. A. Hirsch, *Infinite soluble groups* I, Proc. London Math. Soc. (2) *44* (1938), 53–60.
4. ———, II, *ibid.*, 336–345.
5. ———, III, *ibid.*, *49* (1946), 184–194.
6. S. A. Jennings, *The group ring of a p-group over a modular field*, Trans. Amer. Math. Soc., *50* (1941), 175–185.
7. W. Magnus, *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann., *111* (1935), 259–280.
8. ———, *Ueber Beziehungen zwischen höheren Kommutatoren*, J. reine angew. Math., *177* (1937), 105–115.
9. ———, *Ueber Gruppen und zugeordnete Liesche Ringe*, *ibid.*, *182* (1940), 142–149.
10. A. I. Malcev, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk SSSR Ser. Mat., *13* (1949) 9–32 (AMS Translation No. 39).
11. ———, *Nilpotent torsion-free groups*, *ibid.*, 201–212.

University of British Columbia

GENERALIZED MATRIX ALGEBRAS

W. P. BROWN

1. Introduction. The algebras considered here arose in the investigation of an algebra connected with the orthogonal group.[1] We consider an algebra $\mathfrak{A}$ of dimension mn over a field K of characteristic zero, and possessing a basis $\{e_{ij}\}$ $(1 < i < m;\ 1 < j < n)$ with the multiplication property

$$e_{ij}e_{pq} = \phi_{jp}e_{iq}, \qquad \phi_{jp} \in K. \tag{1}$$

The field elements ϕ_{ij} form a matrix $\Phi = (\phi_{ij})$ of order $n \times m$. It will be called the multiplication matrix of the algebra relative to the basis $\{e_{ij}\}$.

Such algebras will be called generalized matrix algebras. If $m = n$ and $\phi_{ij} = \delta_{ij}$ (the Kronecker delta) we have a total matrix algebra.

An element b of $\mathfrak{A}$ has an expression in terms of the basis $\{e_{ij}\}$ of the form

$$b = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}e_{ij}, \qquad b_{ij} \in K.$$

The correspondence $b \to B = (b_{ij})$ is a one to one correspondence between the elements of $\mathfrak{A}$ and the set of $m \times n$ matrices over K. If $a \in \mathfrak{A}$ and $a \to A = (a_{ij})$, then

$$\begin{aligned} ab &= \sum_{i,j} a_{ij}e_{ij}\sum_{p,q} b_{pq}e_{pq} \\ &= \sum_{i,q}\Big(\sum_{j,p} a_{ij}\phi_{jp}b_{pq}\Big)\, e_{iq}. \end{aligned}$$

It follows that the product ab corresponds to the matrix $A\Phi B$; $ab \to A\Phi B$. The most general change of basis of $\mathfrak{A}$ may be effected by a transformation of the type

$$f_{ij} = \sum_{\lambda=1}^{m}\sum_{\mu=1}^{n} \sigma_{ij}^{\lambda\mu}e_{\lambda\mu}, \qquad \sigma_{ij}^{\lambda\mu} \in K \tag{2}$$

$(1 < i < m;\ 1 < j < n)$. We use double suffix notation with lexicographic ordering to describe the matrix of this transformation. The element of its (ij)th row and $(\lambda\mu)$th column is $\sigma_{ij}^{\lambda\mu}$. The elements f_{ij} of $\mathfrak{A}$ constitute a basis for $\mathfrak{A}$ if and only if this matrix is non-singular. We consider however a special type of transformation, namely:

$$f_{ij} = \sum_{\lambda=1}^{m} s_{i\lambda}e_{\lambda j}, \qquad s_{i\lambda} \in K. \tag{3}$$

This may be written in the form of (2) by setting $\sigma_{ij}^{\lambda\mu} = s_{i\lambda}\delta_{j\mu}$. This is the element of the (ij)th row and $(\lambda\mu)$th column of the Kronecker product $S \times I$ where S is the $m \times m$ matrix (s_{ij}) and I is the $n \times n$ unit matrix. It follows

Received November 2, 1953.

[1]The algebra concerned is ω_f^n. For definition see (1) and (2, chap. V, 5).

that the elements f_{ij} in (3) constitute a basis for $\mathfrak{A}$, if and only if $S = (s_{ij})$ is non-singular.

The nature of the multiplication rule is preserved under transformations of the type occurring in (3). Indeed

$$
\begin{aligned}
f_{ij}f_{pq} &= \sum_{\lambda} s_{i\lambda}e_{\lambda j}\sum_{\mu} s_{p\mu}e_{\mu q} \\
&= \sum_{\lambda,\mu} s_{i\lambda}s_{p\mu}\phi_{j\mu}e_{\lambda q} \\
&= \Big(\sum_{\mu} s_{p\mu}\phi_{j\mu}\Big) f_{iq},
\end{aligned}
$$

by (3). Hence $f_{ij}f_{pq} = \psi_{jp}f_{iq}$ where $\psi_{jp} = \sum_{\mu}\phi_{j\mu}s_{p\mu}$. Relative to the new basis $\{f_{ij}\}$, the algebra has therefore the multiplication matrix $\Psi = \Phi S^T$, where S^T denotes the transpose of S.

We make a further change of basis, namely

$$
g_{ij} = \sum_{\lambda=1}^{n} f_{i\lambda}r_{\lambda j}, \qquad r_{\lambda j} \in K,
$$

$(1 < i < m; 1 < j < n)$. Again the elements g_{ij} of $\mathfrak{A}$ constitute a basis if and only if the $n \times n$ matrix $R = (r_{ij})$ is non-singular. The multiplication rule is again transformed;

$$
g_{ij}g_{pq} = \theta_{jp}g_{iq} \text{ where } \theta_{jp} = \sum_{\lambda} r_{\lambda j}\psi_{\lambda p} \in K.
$$

Relative to the basis $\{g_{ij}\}$ the algebra has the matrix $\Theta = R^T\Phi S^T$.

If Φ has rank r, non-singular matrices R and S may be chosen so that the $n \times m$ matrix Θ is

$$
\Theta = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},
$$

where I_r is the $r \times r$ unit matrix and all other submatrices of Θ are zero. A basis such as $\{g_{ij}\}$, relative to which the multiplication matrix has this simple form, will be called a special basis. While the types of transformation used preserve the rank of the multiplication matrix, it has not yet been demonstrated that the rank of a multiplication matrix is an invariant for $\mathfrak{A}$. This, however, will be obvious later.

2. The structure of generalized matrix algebras. We now assume that a special basis has been chosen for $\mathfrak{A}$ and that the multiplication matrix Φ has the special form of Θ above and has rank r. Suppose that an element $b \in \mathfrak{A}$ has a matrix B whose partitioned form is

$$
B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},
$$

the dimensions of the submatrices being $r \times r$ for B_1, $r \times n - r$ for B_2, $m - r \times r$ for B_3 and $m - r \times n - r$ for B_4.

The radical of A consists of all elements that are properly nilpotent, i.e. all elements b of $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$, ab is nilpotent. The matrix cor-

responding to the tth power of b is $(B\Phi)^{t-1}B$. It follows that b is nilpotent if and only if $B\Phi$ is a nilpotent matrix.

Let a and c be elements of $\mathfrak{A}$ whose matrices are A and C respectively. The matrix of the product abc is the product of matrices $A\Phi B\Phi C$. Now suppose $B_1 = 0$. Then

$$\Phi B\Phi = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

It follows that b is properly nilpotent and has index of nilpotence <3.

On the other hand, for b to be properly nilpotent it is necessary that the matrix $A\Phi B\Phi$ be nilpotent for all matrices A.

$$A\Phi B\Phi = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1B_1 & 0 \\ A_3B_1 & 0 \end{pmatrix}$$

The tth power of this matrix has $(A_1B_1)^t$ in the first submatrix position. It follows that b is properly nilpotent only if A_1B_1 is nilpotent for all $r \times r$ matrices A_1. This can only occur if $B_1 = 0$. Hence the radical of $\mathfrak{A}$ consists of all those elements b whose matrices relative to a special basis, have their first submatrix zero.

A Wedderburn decomposition of $\mathfrak{A}$ into a direct sum of a semisimple subalgebra $\mathfrak{B}$ and the radical $\mathfrak{N}$ is now clear. $\mathfrak{B}$ consists of all elements b of $\mathfrak{A}$ whose matrices relative to a special basis, have the submatrices B_2, B_3 and B_4 all zero. For such elements the mapping $b \to B_1$ is a ring isomorphism of $\mathfrak{B}$ onto the total matrix algebra of degree r over K. Hence $\mathfrak{B}$ is simple and r is an invariant of the algebra.

We see that a generalized matrix algebra is either simple ($m = n =$ rank Φ) or is non semisimple and simple modulo its radical. If the algebra is simple it certainly possesses an identity element. On the other hand let the generalized matrix algebra $\mathfrak{A}$ possess an identity element e so that $ea = ae = a$ for all a in $\mathfrak{A}$. Let E and A be the corresponding matrices. We must have $E\Phi A = A\Phi E = A$ for all matrices A. Hence $E\Phi$ must be the $m \times m$ unit matrix and ΦE must be the $n \times n$ unit matrix. This can happen only if $m = n$ and Φ is non-singular. The algebra is then simple. We restate the above result in a

THEOREM. *A generalized matrix algebra is either*

(i) *simple, or*

(ii) *non-semisimple and simple modulo its radical.*

It is simple if and only if it possesses an identity element.

REFERENCES

1. R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. Math., 38 (1937), 857.
2. H. Weyl, *The classical groups* (Princeton, 1946).

King's College, Aberdeen

SOME THEOREMS ON MATRICES WITH REAL QUATERNION ELEMENTS

N. A. WIEGMANN

1. Introduction. Matrices with real quaternion elements have been dealt with in earlier papers by Wolf **(10)** and Lee **(4)**. In the former, an elementary divisor theory was developed for such matrices by using an isomorphism between $n\times n$ real quaternion matrices and $2n\times 2n$ matrices with complex elements. In the latter, further results were obtained (including, mainly, the transforming of a quaternion matrix into a triangular form under a unitary similarity transformation) by using a different isomorphism. Certain other related results have also been obtained **(1)**. Others, including Moore and Ingraham, have considered quaternion matrices earlier.

The intent here is to consider how other theorems which hold for matrices in the complex field may hold for quaternion matrices. To do this, the isomorphism in **(4)** is employed. First, an analog of the Jordan normal form is obtained; this result is closely related, of course, to the final result in **(10)** concerned with necessary and sufficient conditions for similarity of quaternion matrices, but here a proof is employed which depends entirely on known complex matrix theory, which throws light on the structure of the similarity transformation, and which leads in a natural way to a definition of elementary divisors for quaternion matrices. Next, this Jordan form is used to obtain some results concerning commutative matrices. In part 4, the familiar polar form of a complex matrix is shown to hold in the quaternion case. Next, some further properties of normal quaternion matrices are verified and, in the final section, some properties of quaternion matrices relative to unitary (quaternion) equivalence transformations are obtained.

2. An analog of the Jordan normal form. Let the $n\times n$ quaternion matrix A be written in the form $A = A_1+jA_2$ where A_1 and A_2 are (uniquely determined) matrices with complex elements (where every quaternion element is considered as written in the form $a = (a_1+a_2i)+j(a_3+a_4i)$ where each a_i is real). Form the $2n\times 2n$ complex matrix

$$A^* = \begin{bmatrix} A_1 & -A_2^C \\ A_2 & A_1^C \end{bmatrix}$$

(where A^C denotes the matrix obtained by taking the complex conjugate of each element of A and, later, A^{CT} denotes the transpose of A^C). According to **(4)**, the correspondence between A and A^* is an isomorphism and has properties as developed there.

Received April 30, 1954.

If it is possible to show that for a given A^* there is determined an $n \times n$ matrix J_1 in the (complex) Jordan normal form such that a non-singular matrix P and a matrix J exist so that $A^*P = PJ$ where P and J have the forms, respectively,

$$\begin{bmatrix} P_1 & -P_2^C \\ P_2 & P_1^C \end{bmatrix} \text{ and } \begin{bmatrix} J_1 & 0 \\ 0 & J_1^C \end{bmatrix},$$

then an analog of the Jordan normal form can be obtained for the quaternion matrix A. For it can be easily seen that the inverse of P must also be of the form

$$\begin{bmatrix} Q_1 & -Q_2^C \\ Q_2 & Q_1^C \end{bmatrix},$$

so that $P^{-1}A^*P = J$. But according to the nature of the correspondence between A^* and A this implies that $(Q_1+jQ_2)(A_1+jA_2)(P_1+jP_2) = J_1+j\cdot 0$ or that $(Q_1+jQ_2)A(P_1+jP_2) = J_1$ where $(Q_1+jQ_2)(P_1+jP_2) = I$, so that A is similar to a complex $n \times n$ Jordan form J_1.

Since A^* is a matrix with complex elements, there exists a non-singular matrix P such that $P^{-1}A^*P = J$ is the Jordan form of A^* so that $A^*P = PJ$. In the following steps it will be shown that a P and a J can be obtained which satisfy this relation and are of the desired form.

(a) Let $\alpha_1, \ldots, \alpha_m$ be the distinct (complex) characteristic roots of A^*. Then each column of P is a column vector v with $2n$ elements satisfying one and only one of the following relations:

$$\text{(i) } A^*v = v\alpha_i,$$
$$\text{(ii) } A^*v = w + v\alpha_i,$$

where w is the column vector adjacent to v on the left. All $2n$ column vectors are linearly independent and for each α_i there exists at least one column of the type (i).

(b) For simplicity in notation let α be a root for which $v_1, v_2, \ldots, v_t$ are the set of column vectors of P of type (i) relative to α. Let the column vector v_1^* be defined relative to v_1 as follows: If v_1 is a column vector whose transpose is the row vector

$$[v_{11}, v_{21}, \ldots, v_{n1}, w_{11}, w_{21}, \ldots w_{n1}],$$

then v_1^* is the column vector whose transpose is the row vector

$$[-\bar{w}_{11}, -\bar{w}_{21}, \ldots -\bar{w}_{n1}, \bar{v}_{11}, \bar{v}_{21}, \ldots \bar{v}_{n1}].$$

If v_1 is not the zero vector, then v_1 and v_1^* are linearly independent, for if $c_1v_1 + c_2v_1^* = 0$, it follows that

$$(c_1\bar{c}_1 + c_2\bar{c}_2)w_{k1} = 0, \quad (c_1\bar{c}_1 + c_2\bar{c}_2)v_{k1} = 0 \quad (k = 1, 2, \ldots, n),$$

so that $c_1 = c_2 = 0$. Also, if $A^*v_1 = v_1\alpha$, it follows that $A^*v_1^* = v_1^*\bar{\alpha}$. Let us consider, first, vectors of type (i).

Let α be real. Then v_1 and v_1^* are linearly independent vectors of type (i) for α, and either exhaust the number of such linearly independent column vectors or there exists another, say v_2, which is linearly independent of v_1 and v_1^*. Form v_2^*; then v_1, v_1^*, v_2, and v_2^* are linearly independent, for if $c_1v_1 + c_2v_1^* + c_3v_2 + c_4v_2^* = 0$, then $\bar{c}_1v_1^* - \bar{c}_2v_1 + \bar{c}_3v_2^* - \bar{c}_4v_2 = 0$. But by properly combining these relations, it would follow that

$$(\bar{c}_3c_1 + c_4\bar{c}_2)v_1 + (\bar{c}_3c_2 - c_4\bar{c}_1)v_1^* + (c_3\bar{c}_3 + c_4\bar{c}_4)v_2 = 0,$$

so that $c_3 = c_4 = 0$ and so, from above, $c_1 = c_2 = 0$. Either v_1, v_1^*, v_2, v_2^* exhaust the number of linearly independent vectors of type (i) for α, or they do not. By means of this process there is obtained a set of linearly independent vectors of the form $v_1, v_1^*, \ldots, v_k, v_k^*$ which provide a basis for the vectors of type (i) corresponding to each real α.

Let α be non-real complex. Then if the matrix P contains a set of vectors $v_1, v_2, \ldots, v_t$ such that $A^*v_j = v_j\alpha$ $(j = 1, 2, \ldots, t)$, it follows that $A^*v_j^* = v_j^*\bar{\alpha}$ $(j = 1, 2, \ldots, t)$ (where the v_j^* are linearly independent since the v_j are), that there are no other vectors linearly independent of these for which this is true, and that $\bar{\alpha}$ is also a root of A^*.

Since $\alpha_1, \alpha_2, \ldots, \alpha_m$ are distinct, the sets of linearly independent vectors of type (i) obtained in this way are linearly independent and are equal in number to those column vectors of type (i) in the matrix P.

(c) Consider vectors v of type (ii); these may be written as $(A^* - \alpha_iI)v = w$; and it follows that $(A^* - \bar{\alpha}_iI)v^* = w^*$.

Let α be real. Let there be $2k$ vectors $v_1, v_2, \ldots, v_p, \ldots, v_{2k}$ of P of type (i) corresponding to α and let them be written in such an order that if there exist for some v_i vectors $v_i^{(1)}$ of P so that

$$\text{(iii)} \qquad (A^* - \alpha I)v_i^{(1)} = v_i,$$

these vectors, $v_1, v_2, \ldots, v_p$ are written together and first in this ordering. Then p must be even and a set of linearly independent vectors of the above v, v^* type can be obtained which span the same space as $v_1, v_2, \ldots, v_p$. For if $(A^* - \alpha I)v_1^{(1)} = v_1$, then $(A^* - \alpha I)v_1^{(1)*} = v_1^*$, and since v_1 and v_1^* are linearly independent, $v_1^{(1)}$ and $v_1^{(1)*}$ are also. Either $p = 2$, or the process can be continued as before, so that $p = 2q$. In this way we see that there exists a set of linearly independent vectors, $v_1, v_1^*, \ldots, v_q, v_q^*, \ldots, v_k, v_k^*$ (which form a basis for all vectors of type (i) corresponding to α) such that $v_1^{(1)}$, $v_1^{(1)*}, \ldots, v_q^{(1)}$ $v_q^{(1)*}$, provide a basis for the space spanned by $v_1^{(1)}, \ldots, v_{2q}^{(1)}$ as taken above where v_i and $v_i^{(1)}$ are related as above. If for some of the $v_j^{(1)}$ there exist $v_j^{(2)}$ in P such that

$$(A^* - \alpha I)v_j^{(2)} = v_j^{(1)},$$

the above process can be repeated, and a set of vectors, taken notationally as $v_1^{(2)}, v_1^{(2)*}, \ldots, v_s^{(2)}, v_s^{(2)*}$ (which span the same space as the linearly independent $v_j^{(2)}$), is obtained which stand in the same relation to $v_1^{(1)}, v_1^{(1)*}, \ldots, v_q^{(1)}$, $v_q^{(1)*}$ as the latter do to $v_1, v_1^*, \ldots, v_k, v_k^*$.

If α is non-real complex, let $v_1, v_2, \ldots, v_p, \ldots, v_t$ (the vectors of type (i) corresponding to α) be ordered in such a way that for $v_1, v_2, \ldots, v_p$ there exist $v_1^{(1)}, v_2^{(1)}, \ldots, v_p^{(1)}$ satisfying (iii). Then $(A^* - \bar{\alpha}I)v_i^{(1)*} = v_i^*$, $i = 1,2, \ldots, p$, and $v_1^{(1)*}, v_2^{(1)*}, \ldots, v_p^{(1)*}$ are such that there exists no vector, w, linearly independent of them such that $(A^* - \bar{\alpha}I)w$ is in the space generated by $v_1^*, v_2^*, \ldots, v_p^*$. For some $v_i^{(1)}$ there may exist $v_i^{(2)}$ such that

$$(A^* - \alpha I)v_i^{(2)} = v_i^{(1)};$$

in this way a set of $v_i^{(2)*}$ are determined and the process is seen to be a general one.

(d) By the above, a set of $2n$ linearly independent vectors, taken notationally as $w_1, w_2, \ldots, w_n, w_1^*, w_2^*, \ldots, w_n^*$, are obtained such that either $A^*w_i = w_i\alpha$ (and so $A^*w_i^* = w_i^*\bar{\alpha}$, for any α), or, for some w_i satisfying $A^*w_i = w_i\alpha$, there exist among the above $2n$ vectors certain vectors taken notationally as $w_i^{(1)}, w_i^{(2)}, \ldots, w_i^{(s)}$ such that

$$\begin{aligned} (A^* - \alpha I)w_i^{(1)} &= w_i, \\ (A^* - \alpha I)w_i^{(j)} &= w_i^{(j-1)}, \qquad j = 2, 3, \ldots, s; \end{aligned}$$

in this case it follows that

$$\begin{aligned} (A^* - \bar{\alpha} I)w_i^{(1)*} &= w_i^*, \\ (A^* - \bar{\alpha} I)w_i^{(j)*} &= w_i^{(j-1)*}, \qquad j = 2, 3, \ldots, s. \end{aligned}$$

It is now evident that by properly arranging the w_i and w_i^*, a $2n \times 2n$ matrix P can be obtained such that $A^*P = PJ$ as indicated above. If in J_1 (as used there), the roots $\alpha = a + bi$ are such that $b \geqq 0$, then a canonical form has been obtained for A^* and hence for the quaternion matrix A.

THEOREM 1. *Every $n \times n$ matrix with real quaternion elements is similar under a matrix transformation with real quaternion elements to a matrix in (complex) Jordan normal form with diagonal elements of the form $a + bi$, $b \geqq 0$.*

3. Properties of commutative matrices. According to a theorem due to Taber **(5)**, if a matrix A with complex elements is non-derogatory, the only matrices commutative with A are polynomial functions of A. An equivalent theorem had been previously given by Frobenius **(3**, Theorem XIII**)**.

In order to obtain an analog for this theorem where A contains real quaternion elements, let such a matrix A be defined to be non-derogatory when its Jordan normal form matrix (as obtained in the preceding) is non-derogatory.

Let A and B be quaternion matrices such that $AB = BA$ where A is non-derogatory. Let $PAP^{-1} = J = J_1 \dotplus J_2 \dotplus \ldots \dotplus J_m$ be the Jordan form of A where:

$$J_i = \begin{bmatrix} \alpha_i & 1 & 0 & \ldots & 0 \\ 0 & \alpha_i & 1 & \ldots & 0 \\ & \ldots & & \ldots & \\ 0 & \ldots & & \alpha_i & 1 \\ 0 & \ldots & & 0 & \alpha_i \end{bmatrix}, \qquad i = 1,2, \ldots, m,$$

where $\alpha_i \neq \alpha_j$ when $i \neq j$ and $\alpha_k = a_k + ib_k$, $b_k \geq 0$. Let $PBP^{-1} = B_1$ so that $JB_1 = B_1J$.

LEMMA. *$B_1 = B_{11} \dotplus B_{12} \dotplus \ldots \dotplus B_{1m}$ where B_{1i} has the same order as J_i, where*

$$B_{1i} = \begin{bmatrix} b_{i1} & b_{i2} & b_{i3} & & \cdots & & b_{in} \\ 0 & b_{i1} & b_{i2} & & \cdots & & \\ 0 & 0 & b_{i1} & & \cdots & & \\ & \cdots & & & \cdots & & \\ & \cdots & & b_{i1} & & b_{i2} & b_{i3} \\ & \cdots & & 0 & & b_{i1} & b_{i2} \\ 0 & \cdots & & 0 & & 0 & b_{i1} \end{bmatrix}$$

and where

(i) *if α_i is real, the non-diagonal elements of B_{1i} are quaternions while the diagonal elements are complex;*

(ii) *if α_i is non-real complex, the elements of the corresponding B_{1i} are complex.*

The following may be noted: if α is a non-real complex element, b a quaternion element, and $\alpha b = b\alpha$, then b is a complex number; if α is non-real complex, if β is complex, and c a quaternion element such that $\alpha c = c\alpha + \beta$, then c is complex and $\beta = 0$. Also no α_i, above, is the conjugate of an α_j.

Let $B_1 = (b_{ij})$, let J_1 be of order $r \times r$, and consider the upper left $r \times r$ sub-matrix of the product $JB_1 = B_1J$. The following relations result:

$$\begin{aligned} &\alpha_1 b_{r1} = b_{r1}\alpha_1, && \\ &\alpha_1 b_{ri} = b_{r,i-1} + b_{ri}\alpha_1, && i = 2, 3, \ldots, r, \\ &\alpha_1 b_{i1} + b_{i+1,1} = b_{i1}\alpha_1, && i = 1, 2, \ldots, r-1, \\ &\alpha_1 b_{it} + b_{i+1,t} = b_{t,t-1} + b_{it}\alpha_1, && \begin{cases} t = 2, 3, \ldots, r, \\ i = 1, 2, \ldots, r-1. \end{cases} \end{aligned}$$

If α_1 is real then, although the b_{ij} are quaternion elements, all commutative properties hold for these relations (as in the complex case as treated by Taber) and the upper left $r \times r$ matrix has the form B_{11} with all quaternion elements, in general. If α_1 is non-real complex, it follows from the first relation that b_{r1} is complex; from this and the third relation it follows that all elements in the first column above b_{r1} are complex (and in fact, except for b_{11}, all are 0); from the second relation it can be seen that all elements of the rth row of this submatrix are 0 except b_{rr} which is complex. Using the fourth set of relations, we see that the remaining elements are complex, all necessary commutative properties hold, and that the submatrix has the B_{11} form. B_{11} now has the required form unless, for a real α_1, the diagonal elements are quaternions; if so, there exists a quaternion element b such that $bb_{i1}\bar{b} = \beta$ is a complex number where $b\bar{b} = 1$. Form the $n \times n$ matrix $Q = bI_1 \dotplus I_2$ where I_1 and I_2 are identity matrices and I_1 is of order $r \times r$. Then $Q^{-1} = \bar{b}I_1 \dotplus I_2$ and QB_1Q^{-1} has the form required and $QJQ^{-1} = J$.

Let J_2 be of order $s \times s$, and consider the $s \times r$ submatrix directly below B_{11} in the matrix B_1. Upon comparing corresponding elements of this $s \times r$ submatrix in the product $JB_1 = B_1J$, we see that the set of following relations appear:

$$\begin{array}{ll}
\alpha_2 b_{r+s,1} = b_{r+s,1}\alpha_1, & \\
\alpha_2 b_{r+s,i} = b_{r+s,i-1} + b_{r+s,i}\alpha_1, & i = 2, \ldots, r, \\
\alpha_2 b_{i1} + b_{i+1,1} = b_{i1}\alpha_1, & i = r+1, \ldots, r+s-1, \\
\alpha_2 b_{it} + b_{i+1,t} = b_{i,t-1} + b_{it}\alpha_1, & \begin{cases} t = 2, 3, \ldots, r, \\ i = r+1, r+2, \ldots r+s-1. \end{cases}
\end{array}$$

Since, for $i \neq j$, $\alpha_i \neq \alpha_j$ and $\alpha_i \neq \bar{\alpha}_j$, it follows from these relations that all elements of this $s \times r$ submatrix of B_1 are zero. In this way it can be shown that $B_1 = B_{11} \dotplus B_2$ where B_{11} has the form given in the lemma. When B_2 is treated in like fashion, the lemma follows.

Consider next the possibility of representing this B_1 as a polynomial in J_1 where J_1 contains only complex elements. It is evident (from the work of Taber or by merely considering the set of equations obtained) that it is possible to determine two sets, x_i and x_i', $i = 0, 1, 2, \ldots, n-1$, of quaternion elements such that

$$B_1 = \sum_{i=0}^{n-1} x_i J^i = \sum_{i=0}^{n-1} J^i x_i'.$$

If all the diagonal elements of J are real, $x_iJ^i = J^ix_i$; if all the diagonal elements of J are non-real complex, all elements of B_1 are complex and so are the x_i so that again $x_iJ^i = J^ix_i$; and the same would be true if all the elements of B_1 were complex regardless of the nature of the α_i in J. In these instances if $x_j = \rho_j u_j$ (where ρ_j is the real absolute value of the quaternion element x_j and u_j the related quaternion of absolute value one), then

$$B = P^{-1}B_1P = \sum_{i=0}^{n-1} \rho_i P^{-1}(u_iI)P \cdot P^{-1}(J^i)P = \sum_{i=0}^{n-1} \rho_i U_i A^i$$

where $U_i = P^{-1}(u_iI)P$ and $U_iA = AU_i$ for each i. It follows that:

THEOREM 2. *If A and B are quaternion matrices, if $AB = BA$, and if A is non-derogatory with either all real or all non-real complex roots, then*

$$B = \sum_{i=0}^{n-1} \rho_i U_i A^i$$

where the ρ_i are real, $U_iA = AU_i$ for each i, and each U_i has a single characteristic root of absolute value one.

4. A polar form. Every complex number has the familiar polar form $\rho e^{i\theta}$ and, as has been seen, the same is true for a quaternion. For a matrix A with complex elements a polar representation has been obtained when A is non-singular by Wintner and Murnaghan **(9)** and when A is singular by Williamson **(7)**. It also exists for quaternion matrices according to the following:

THEOREM 3. *Every $n \times n$ matrix A with real quaternion elements can be expressed as $A = H_1W_1 = W_1K_1$ where H_1 and K_1 are hermitian (quaternion) matrices and W_1 is a unitary (quaternion) matrix; if A is non-singular the representation is unique, and if A is singular, H_1 and K_1 are unique but W_1 is arbitrary to some extent.*

Let $A = A_1 + jA_2$ where A_1 and A_2 are (as in §2) uniquely determined matrices with complex elements. Then A is isomorphic to A^* where

(i) $$A^* = \begin{bmatrix} A_1 & -A_2^C \\ A_2 & A_1^C \end{bmatrix}.$$

Since A^* has complex elements $A^* = HU = UK$ by **(9)** and **(7)**, where H and K are hermitian and U unitary. Then AA^{CT} corresponds to H^2 and there exists a unitary quaternion matrix (see **(4)**, for example) $V_3 = V_1 + jV_2$ so that $V_3AA^{CT}V_3^{CT} = D$ is a diagonal matrix with real elements and, consequently, if

$$V = \begin{bmatrix} V_1 & -V_2^C \\ V_2 & V_1^C \end{bmatrix},$$

then $VA^*A^{*CT}V^{CT} = D \dotplus D$. Since H is hermitian with non-negative real roots, there exists a unitary matrix W such that $WHW^{CT} = D_1$ is diagonal with these non-negative real roots along the diagonal; and this W can be chosen in such a way that

$$WH^2W^{CT} = WA^*A^{*CT}W^{CT} = D \dotplus D$$

so that $D_1^2 = D \dotplus D$ and so $D_1 = D_2 \dotplus D_2$ where the diagonal elements of D_2 are the positive square roots of the corresponding real roots of D. Then H must be of the same form as A^* in (i) for if $X = VW^{CT}$, then

$$XWA^*A^{*CT}W^{CT}X^{CT} = X(D \dotplus D)X^{CT} = VA^*A^{*CT}V^{CT} = D \dotplus D$$

so that $X(D \dotplus D) = (D \dotplus D)X$ and so $X(D_2 \dotplus D_2) = (D_2 \dotplus D_2)X$. From this, $XWHW^{CT}X^{CT} = XD_1X^{CT} = D_1 = VHV^{CT}$ so that $H = V^{CT}D_1V$ and from the form of the matrices on the right side of this equality, their product is of type (i).

From $A^* = HU$, it follows that $VA^*V^{CT} = VHV^{CT}VUV^{CT}$ where the matrices have the form

$$\begin{bmatrix} B_1 & -B_2^C \\ B_2 & B_1^C \end{bmatrix} = \begin{bmatrix} D_2 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}.$$

If A^* is non-singular, D_2 is non-singular and by equating corresponding block matrices, $U_4 = U_1^C$ and $U_2 = -U_3^C$.

If A^* is singular (in which case there is some arbitrariness involved in the choice of U in HU), then D_2 is singular; let the first r diagonal elements be non-zero, the remaining being 0. From this it can be seen that $D_2(U_1 - U_4^C) = 0$ and $D_2(U_3 + U_2^C) = 0$; this means that the first r rows of U_4 are the

conjugates of the first r rows of U_1 and the first r rows of U_2 are the negative conjugates of the first r rows of U_3. Since VUV^{CT} is unitary, these $2r$ rows are linearly independent and by means of the v, v^*-basis procedure employed in §2 above, it is seen that it is possible to complete the remaining rows of this matrix so that it is unitary and of the form (i). From the form of each matrix in the $2n \times 2n$ matrix relation $A^* = HU$, it follows that A can be expressed as required by the theorem. Since $U^{CT}A^* = U^{CT}HU = K$ is hermitian, $A^* = UK$ holds (uniquely if A^* is non-singular) and the theorem is true.

5. Properties of normal quaternion matrices. If A is a normal quaternion matrix, it can be brought into diagonal form under a unitary similarity transformation (see **(4)**, for example). Some further properties of normal quaternion matrices are verified here.

It is known that a complex matrix A is normal if and only if A^{CT} is a polynomial in A. If A is a normal quaternion matrix, there exists a unitary quaternion matrix U such that $UAU^{CT} = D$ where the characteristic roots of A appear in the diagonal matrix D. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the distinct roots of A, the set of equations

$$\bar{\alpha}_i = \sum_{j=0}^{m-1} x_j \alpha_i^{\,j}, \qquad i = 1, 2, \ldots, m,$$

in x_j always have solutions in the complex field. This implies that

$$D^{CT} = \sum_{j=0}^{m-1} x_j D^j$$

and, if $x_j = \rho_j \cdot e^{i\theta_j}$,

$$A^{CT} = \sum_{j=0}^{m-1} \rho_j U(e^{i\theta_j} I)\, U^{CT} A^j = \sum_{j=0}^{m-1} \rho_j V_j A^j$$

where $V_j = U(e^{i\theta_j} I)U^{CT}$ is unitary and $V_j A = A V_j$ for all j. If more latitude is allowed for the degree of the polynomial, let the distinct roots be written in the form $\alpha_1, \alpha_2, \ldots, \alpha_r, \ldots, \alpha_m$ where $\alpha_1, \ldots, \alpha_r$ are the non-real complex roots. Let the roots $\alpha_1, \alpha_2, \ldots, \alpha_r, \ldots \alpha_m, \bar{\alpha}_1, \ldots, \bar{\alpha}_r$ be used to form the $m + r$ equations

$$\bar{\beta}_i = \sum_{j=0}^{m+r-1} x_j \beta_i^{\,j}$$

where β_i runs through the latter set of α_i and $\bar{\alpha}_i$; in this case the x_j will all be real and it follows that:

THEOREM 4. *A quaternion matrix A is normal if and only if A^{CT} is a polynomial in A with real coefficients.*

The following theorem will now be shown to hold as in the complex case:

THEOREM 5. *Two normal quaternion matrices A and B are commutative if and only if they can be diagonalized by the same unitary transformation.*

If $AB = BA$, let $UAU^{CT} = D$ where D is diagonal such that like roots are in consecutive order, with real roots $\alpha_1, \ldots, \alpha_s$ first, and complex roots $\beta_1, \ldots, \beta_t$, $(\beta_k = \gamma_k + i\delta_k, \delta_k > 0)$ next. Let

$$\begin{bmatrix} D & 0 \\ 0 & D^C \end{bmatrix} \text{ and } \begin{bmatrix} C_1 & -C_2{}^C \\ C_2 & C_1{}^C \end{bmatrix}$$

be the $2n \times 2n$ complex matrices which are isomorphic to D and UBU^{CT}, respectively. From the commutative property $DC_1 = C_1D$ and $D^CC_2 = C_2D$, and so

$$\begin{aligned} C_1 &= C_{11} \dotplus \ldots \dotplus C_{1s} \dotplus C'_{11} \dotplus \ldots \dotplus C'_{1t}, \\ C_2 &= C_{21} \dotplus \ldots \dotplus C_{2s} \dotplus 0 \quad \dotplus \ldots \dotplus 0, \end{aligned}$$

where $D = \alpha_1 I_1 \dotplus \ldots \dotplus \alpha_s I_s \dotplus \beta_1 I'_1 \dotplus \ldots \dotplus \beta_t I'_t$ and where C_{1j} and C_{2j} have the same order as the identity matrix I_j and C'_{1j} and the corresponding 0 matrix in C_2 have the same order as the identity matrix I'_j. Therefore,

$$\begin{aligned} UBU^{CT} &= (C_{11}\dotplus\ldots\dotplus C_{1s}\dotplus C'_{11}\dotplus\ldots\dotplus C'_{1t}) + j(C_{21}\dotplus\ldots\dotplus C_{2s}\dotplus 0\dotplus\ldots\dotplus 0) \\ &= (C_{11}+jC_{21}) \dotplus \ldots \dotplus (C_{1s}+C_{2s}) \dotplus C'_{11} \dotplus \ldots \dotplus C'_{1t} \end{aligned}$$

where the C'_{1j} have only complex elements. Since UBU^{CT} is normal, so is each matrix in the above direct sum; there exist, then, unitary quaternion matrices W_k which diagonalize $C_{1k} + jC_{2k}$ and unitary complex matrices V_k which diagonalize C'_{1k}, for all the above k. If V is the unitary matrix formed by taking the appropriate direct sum of these W_k and V_k, it follows that $VUBU^{CT}V^{CT}$ is diagonal and that $VUAU^{CT}V^{CT} = VDV^{CT} = D$ is also diagonal. The converse is immediate.

The above generalizes as in the complex case:

THEOREM 6. *If $\{A_i\}$ is a set of normal quaternion matrices which commute in pairs, they can be diagonalized by the same unitary transformation.*

If each of the A_i have a single characteristic root, α_i, the theorem is true. If these roots are all real, the theorem is trivially true. If at least one root, say α_k, is non-real complex, let $VA_kV^{CT} = \alpha_k I$ and $VA_iV^{CT} = A'_i$ for all other i; then each A'_i commutes with $\alpha_k I$ and so all A'_i are normal, complex, and commutative in pairs, and can all be diagonalized by a complex unitary matrix U. Therefore, the unitary matrix UV diagonalizes all A_i.

In general, the proof follows by induction on the order of the A_i. The theorem is trivially true for 1×1 matrices. Assume the theorem to be true for $(n-1) \times (n-1)$ matrices. It may also be assumed that there is at least one matrix, A_j, which has at least two distinct roots; let $UA_jU^{CT} = D$ be diagonal (in the same form as D in the preceding theorem). Then each UA_iU^{CT} commutes with D, the problem is reduced to that involving matrices of order less than n and the theorem is true.

The following theorems are true in the complex case **(6)**; they are also true (obviously so from the isomorphism above) in the quaternion case:

THEOREM 7. *A quaternion matrix A is normal if and only if its polar matrices commute.*

THEOREM 8. *If A, B and AB are normal quaternion matrices, then BA is normal.*

THEOREM 9. *If A and B are normal quaternion matrices, then AB is normal if and only if each of A and B commutes with the hermitian polar matrix of the other.*

6. A diagonal form under unitary equivalence transformations. It is also possible to bring a quaternion matrix into a real diagonal matrix under a unitary equivalence transformation according to the following:

THEOREM 10. *For every $r \times s$ quaternion matrix A there exist two unitary quaternion matrices U and V (of dimensions $r \times r$ and $s \times s$, respectively) such that $UAV = D$ is diagonal with non-negative real roots along the diagonal.*

Let $A = A_1 + jA_2$ where A_1 and A_2 are complex, as before, but $r \times s$ in dimension. Let C be the $2r \times 2s$ matrix (composed of A_1 and A_2) with complex elements which corresponds to A. According to a corollary due to Eckert and Young **(2)**, if U is a $2r \times 2r$ unitary matrix which diagonalizes CC^{CT}, there exists a $2s \times 2s$ unitary matrix V such that $UCV = D_1$ is a $2r \times 2s$ diagonal matrix with non-negative real elements. From preceding work, this U may be taken as being in the form

$$\begin{bmatrix} U_1 & -U_2^C \\ U_2 & U_1^C \end{bmatrix},$$

so that $UCC^{CT}U^{CT} = D_2 \dotplus D_2$ is $2r \times 2r$ and so $UCV = D \dotplus D$ where D is $r \times s$, where the elements are non-negative real, and where $(D \dotplus D)(D \dotplus D)^{CT} = D_2 \dotplus D_2$. It remains to verify that V has the proper structure (i.e., like that of U). By considering the relation $UC = (D \dotplus D)V^{CT}$, it follows (as in the proof of the polar representation above) that V has this form where some arbitrariness may be involved, as before, in choosing V. If the components of V are V_1 and V_2, then $(U_1 + jU_2)A(V_1 + jV_2) = D$ as required in the theorem.

As in the complex case **(8)**, it is also true that

THEOREM 11. *If A and B are two $r \times s$ quaternion matrices, then there exist two unitary quaternion matrices U and V such that $UAV = D_1$ and $UBV = D_2$ are complex diagonal matrices if and only if AB^{CT} and $B^{CT}A$ are normal matrices.*

If such a U and V exist, the theorem is obviously true.

If AB^{CT} and $B^{CT}A$ are normal, but the preceding theorem $U_1AV_1 = D_1$ is a non-negative real diagonal matrix and $U_1BV_1 = C$. Let

$$D_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where D is non-singular and B_1 has the same order as D. From the given conditions D_1C^{CT} and $C^{CT}D_1$ are normal; using the former, it follows that $(B_3D)(B_3D)^{CT} = 0$ (where B_3 has quaternion elements and D is real) so that $B_3D = 0$ and so $B_3 = 0$. Similarly $B_2 = 0$. Therefore $DB_1{}^{CT}$ and $B_1{}^{CT}D$ are normal. Now the characteristic roots of $DB_1{}^{CT}$ and $B_1{}^{CT}D$ are the same. (In the complex case, the characteristic roots of MN are the same as those of NM; from the isomorphism used above between $n \times n$ quaternion matrices and $2n \times 2n$ complex matrices, this result is seen to carry over). Therefore, from §5, there exists a polynomial $f(x)$ with real coefficients such that $B_1D = f(DB_1{}^{CT})$ and $DB_1 = f(B_1{}^{CT}D)$ and so $DB_1 = f(B_1{}^{CT}D) = D^{-1}f(DB_1{}^{CT})D = D^{-1}B_1DD$ or $D^2B_1 = B_1D^2$. Since D has positive diagonal elements, $DB_1 = B_1D$. Since $DB_1{}^{CT}B_1D = B_1D \cdot DB_1{}^{CT}$, then $B_1{}^{CT}B_1 = B_1B_1{}^{CT}$ and B_1 is a normal quaternion matrix which commutes with the (normal) real diagonal matrix D. There exists a quaternion unitary matrix W_1 which diagonalizes each simultaneously; there also exist unitary matrices W_2 and W_3 so that $W_2B_4W_3$ is a real diagonal matrix. By multiplying D_1 and C_1 each on the left and right, respectively, by the matrices

$$\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \quad \begin{bmatrix} W_1{}^{CT} & 0 \\ 0 & W_3 \end{bmatrix},$$

the theorem follows.

References

1. J. L. Brenner, *Matrices of quaternions*, Pac. J. Math., *1* (1951), 229–335.
2. C. Eckart and G. Young, *A principal axis transformation for non-hermitian matrices*, Bull. Amer. Math. Soc., *45* (1939), 118–121.
3. G. Frobenius, *Ueber lineare Substitutionen und bilineare Formen*, J. reine angew. Math., *84* (1878), 1–63.
4. H. C. Lee, *Eigenvalues and canonical forms of matrices with quaternion coefficients*, Proc. Royal Irish Academy, *52*, Section A, no. 117 (1949), 253–260.
5. H. Taber, *On the matrix equation $\phi\Omega = \Omega\phi$*, Proc. Amer. Acad. Arts. Sci., *26* (1890–1891), 64–66; *On a theorem of Sylvester's relating to non-degenerate matrices*, *27* (1891–1892), 46–56.
6. N. Wiegmann, *Normal products of matrices*, Duke Math. J., *15* (1948), 633–638.
7. J. Williamson, *A polar representation of singular matrices*, Bull. Amer. Math. Soc., *41* (1935), 118–123.
8. J. Williamson, *Note on a principal axis transformation for non-hermitian matrices*, Bull. Amer. Math., Soc., *45* (1939), 920–922.
9. A. Wintner and F. Murnaghan, *On a polar representation of non-singular square matrices*, Proc. Nat. Acad. Sci., U.S.A., *17* (1931), 676–678.
10. L. A. Wolf, *Similarity of matrices in which the elements are real quaternions*, Bull. Amer. Math. Soc., *42* (1936), 737–743.

Catholic University,
Washington, D.C.

A THEOREM CONCERNING THREE FIELDS

I. N. HERSTEIN

Several authors (**1**; **2**; **3**; **4**; **5**; **6**) have recently studied the existence and non-existence of certain types of extensions of a given field. In this note we prove a theorem closely related to these results which, in a sense, contains essential portions of each of these. We prove the

THEOREM. *Let $F \subset K \subset L$ be three fields (where we assume these inclusions all to be proper). Suppose that for every element x in L there exists a nontrivial polynomial $f_x(t)$ in the variable t with coefficients in F (and which depend on x) such that the element $f_x(x)$ is in K. Then either*

(a) *L is purely inseparable over K, or*

(b) *L, and so K, is algebraic over F.*

Proof. Suppose that L is not purely inseparable over K. Then there exists an element in L which is not in K which is separable over K. The set of all elements in L which are separable over K form a subfield L' of L. K is of course contained in L'; by supposing that L was not purely inseparable over K we have that $L' \neq K$. If this subfield L' were algebraic over F, then K would also be algebraic over F. This, combined with the fact that L is algebraic over K, would then lead to the desired conclusion that L is algebraic over F. So we suppose, to the contrary, that there is some element $a \in L'$, $a \notin K$ which is transcendental over F. (Being in L', a is of course separable over K.) We shall show that this leads to a contradiction.

Let $\bar{L} = F(a)$, the set of all rational functions in a over the field F. Let $\bar{K} = \bar{L} \cap K$. Consider the three fields $F \subset \bar{K} \subset \bar{L}$. These inclusions are all proper since $a \in \bar{L}$, $a \notin \bar{K}$, and since a is algebraic over $\bar{K}$ but not over F. Also, if $x \in \bar{L}$ then there is a polynomial $f_x(t)$ with coefficients in F so that $f_x(x) \in K$; since $f_x(x) \in \bar{L}$ it follows that $f_x(x) \in \bar{K}$. Thus the conditions on the three fields F, K, L carry over to the three fields $F, \bar{K}, \bar{L}$.

By Lüroth's theorem $\bar{K}$ is a rational function field over F in some s, $\bar{K} = F(s)$. $\bar{L} = \bar{K}(a)$ is of finite degree and separable over $\bar{K}$. Now Nagata, Nakayama and Tuzuku (**5**) have proved for this situation that there exist two distinct logarithmic valuations V_1 and V_2 on $\bar{L}$ which coincide on $\bar{K}$; a simple modification of their argument yields that we can find such V_1 and V_2 which, in addition, are trivial on the field F. Thus for these two valuations we have the following properties:

(1) There exists a $u \in \bar{L}$, $u \notin \bar{K}$ so that $V_1(u) \neq V_2(u)$;

(2) $V_1(k) = V_2(k)$ for all $k \in \bar{K}$;

(3) $V_1(\alpha) = V_2(\alpha) = 0$ for all $\alpha \neq 0 \in F$.

Received August 12, 1954. The author is greatly indebted to Professor Irving Kaplansky for his suggestions in the formulation in its present form of the theorem proved in this note.

Without loss of generality we may assume that $V_1(u) > 0$. By hypothesis, $k = u^n + \alpha_{n-1}u^{n-1} + \ldots + \alpha_r u^r \in \bar{K}$ for some $\alpha_i \in F$ $\alpha_r \neq 0$, $n > r > 1$. Thus $V_1(k) = V_2(k)$.

Since $V_1(\alpha_i) = 0$ (we only consider the non-zero multipliers that occur in the expression for k) and since $\alpha_r \neq 0$, $V_1(\alpha_r u^r) = rV_1(u) < V_1(\alpha_m u^m) = mV_1(u)$ for $m > r$ occurring in the expression for k with non-zero multiplier. Thus, since V_1 is a non-Archimedean valuation, it follows that $V_1(k) = rV_1(u)$. Since $0 < V_1(k) = V_2(k)$, it follows that $V_2(u) > 0$. Thus the argument used above for V_1 can be repeated and it follows that $V_2(k) = rV_2(u)$. But $V_1(k) = V_2(k)$; therefore we are led to $rV_1(u) = rV_2(u)$, which, since $r \neq 0$ implies that $V_1(u) = V_2(u)$. This is contrary to the assumption that $V_1(u) \neq V_2(u)$. The theorem is thereby established.

References

1. I. N. Herstein, *The structure of a certain class of rings*, Amer. J. Math., *75* (1953), 864–871.
2. M. Ikeda, *On a theorem of Kaplansky*, Osaka Math. J., *4* (1952), 235–240.
3. I. Kaplansky, *A theorem on division rings*, Can. J. Math., *3* (1951), 290–292.
4. M. Krasner, *The non-existence of certain extensions*, Amer. J. Math., *75* (1953), 112–116.
5. M. Nagata, T. Nakayama, and T. Tuzuku, *On an existence lemma in valuation theory*, Nagoya Math. J., *6* (1953), 59–61.
6. T. Nakayama, *The commutativity of division rings*, Can. J. Math., *5* (1953), 242–244.

University of Pennsylvania

ON ALGEBRAIC SURFACES TERMWISE INVARIANT UNDER CYCLIC COLLINEATIONS

J. C. MORELOCK AND N. C. PERRY

1. Introduction. In algebraic geometry it is of interest to examine polynomial surfaces F which transform into themselves under the collineation T defined by:

$$(x'_1, x'_2, x'_3, x'_4) = (x_1, Ex_2, E^2x_3, E^3x_4)$$

where $E^p = 1$, and p is a prime **(2)**. One of the most obvious ways to ensure invariance of a surface is for each term $x_1^ax_2^bx_3^cx_4^d$ of F to go into itself. We present initially, therefore, a theorem which will be useful in the study of such termwise invariance for polynomials of composite degree. Specializations of this result are then developed for the case when the polynomial degree is a prime.

2. Polynomial surfaces of composite degree. Assume that each term $x_1^ax_2^bx_3^cx_4^d$ of the polynomial is invariant and of degree mp. Then since the transform of this term under T is $x_1^ax_2^bx_3^cx_4^d(E^bE^{2c}E^{3d})$ there must hold the simultaneous equations $b + 2c + 3d = kp$ and $a + b + c + d = mp$. Diophantine solutions in terms of two parameters n and r are readily developed as

$$a = n, \quad b = r - 2n + (2m - k)p, \quad c = n - 2r + (k - m)p, \quad d = r.$$

Hence, in terms of congruence classes with respect to the modulus p, $a \in (n)$, $b \in (r - 2n)$, $c \in (n - 2r)$, $d \in (r)$.

On the other hand let a, b, c and d (which are such that $a + b + c + d = mp$) be assumed to have membership in the classes (n), $(r - 2n)$, $(n - 2r)$, (r). Then the term $x_1^ax_2^bx_3^cx_4^d$ transforms under T into $x_1^ax_2^bE^bx_3^cE^{2c}x_4^dE^{3d}$. Since

$$b + 2c + 3d \equiv (r - 2n) - (2n - 4r) + (3r) \equiv 0 \pmod{p},$$

and $E^{kp} = 1$, invariance is established. Thus we have

THEOREM 1. *A polynomial term $x_1^ax_2^bx_3^cx_4^d$, which has degree a multiple of a prime, p, will go into itself under T if and only if a, b, c, d are respectively members of the congruence classes* (mod p)*:* (n), $(r - 2n)$, $(n - 2r)$, (r).

3. Polynomial surfaces of prime degree. Of particular interest in algebraic geometry are surfaces where the degree of the defining polynomial is not a composite integer but is exactly equal to a prime $p > 3$. For this case n and r are subject to certain restrictions which we now study by setting $m = 1$ in the solution to the simultaneous equations used to develop Theorem 1.

Since a, b, c, d are non-negative exponents where each is at most equal to p, it is clear on inspecting the equation $b + 2c + 3d = kp$, that k can take on only the values 0, 1, 2, and 3.

If $k = 0$, $b = c = r = 0$, and $n = p$.

If $k = 1$, we have the four inequalities:

$$\begin{aligned} &(1) \quad 0 < n, \\ &(2) \quad 0 < r - 2n + p, \\ &(3) \quad 0 < n - 2r, \\ &(4) \quad 0 < r. \end{aligned}$$

From inequality (3) we have $2r < n$, and from (2) it follows that $2n < r + p$, or $n < \frac{1}{2}(p + r)$. Thus n must lie within the range defined by $2r < n < \frac{1}{2}(p+r)$.

Since $4r < 2n$ we may replace $2n$ by $4r$ in the inequality (2) obtaining $3r < p$, or $r < \frac{1}{3}p$. Hence, to obtain an invariant term of degree p, necessary restrictions on r and n are (for the case $k = 1$) that r first be chosen on the range $0 < r < \frac{1}{3}p$, and that then n be chosen on the range $2r < n < \frac{1}{2}(p + r)$.

A similar analysis for the case when $k = 2$ demonstrates that r be chosen on the range $0 < r < \frac{2}{3}p$, and then n be chosen on the range $2r - p < n < \frac{1}{2}r$. An inspection of the latter inequality shows that n is non-negative only if $r > \frac{1}{2}(p + 1)$. This suggests splitting the range of r obtained into the two ranges

$$0 < r < \tfrac{1}{2}p \quad \text{and} \quad \tfrac{1}{2}(p + 1) < r < \tfrac{2}{3}p,$$

with corresponding ranges for n of

$$0 < n < \tfrac{1}{2}r \quad \text{and} \quad 2r - p < n < \tfrac{1}{2}r.$$

Finally, then, if $k = 3$, $b + 2c + 3d =$ p, hence $r = p$ and $n = b = c = 0$.

To summarize, restrictions on n and r necessary for the degree of the polynomial to be precisely p are:

$$\begin{array}{lll} (k = 0) & \text{(i)} & r = 0,\ n = p; \\ (k = 1) & \text{(ii)} & 0 < r < \frac{1}{3}p;\ 2r < n < \frac{1}{2}(p + r); \\ (k = 2) & \text{(iiia)} & 0 < r < \frac{1}{2}p;\ 0 < n < \frac{1}{3}r; \\ & \text{(iiib)} & \frac{1}{2}(p + 1) < r < \frac{2}{3}p;\ 2r - p < n < \frac{1}{2}r; \\ (k = 3) & \text{(iv)} & r = p,\ n = 0. \end{array}$$

We must now demonstrate the sufficiency of the above inequalities, i.e., that any n and r so chosen will yield a polynomial term of degree p. We illustrate the method of proof for (ii), for the case where p is of the form $6\alpha + 1$ and K is any even member of the range $0 < r < \frac{1}{3}p$. L is any corresponding member of the range $2r < n < \frac{1}{2}(p + r)$. Then

$$r = \tfrac{1}{3}(p - 3K - 1), \quad 0 < K < \tfrac{1}{3}(p - 1)$$

and

$$n = \tfrac{1}{2}(p + r - 2L - 1), \quad 0 < L < \tfrac{3}{2}K.$$

The exponent $b = r - 2n + p$ is equal to $2L + 1$ upon substitution and simplification, and in the same way the exponent $c = n - 2r$ can be shown to equal $(3/2)K - L$. Then the polynomial term corresponding to the particular choice of K and L will be

$$x_1^{\frac{1}{2}(p+r-2L-1)} x_2^{2L+1} x_3^{(3/2)K-L} x_4^{\frac{1}{2}(p-3K-1)}.$$

The degree of this term is p, as can be readily shown by adding the exponents.

In a like manner the sufficiency of (ii) for $p = 6\alpha + 1$ and K odd can be readily established. A similar treatment will establish the inequalities (iiia) and (iiib) for all possible cases where p is either of the form $6\alpha + 1$ or $6\alpha - 1$ and K is either even or odd. Since conditions (i) and (iv) are obvious, sufficiency is completely demonstrated. We summarize these results in the following:

THEOREM 2. *Necessary and sufficient conditions that a polynomial term $x_1^a x_2^b x_3^c x_4^d$ of prime degree $p > 3$ shall be invariant under T are that:*

$$a \in (n), \quad b \in (r - 2n), \quad c \in (n - 2r), \quad d \in (r),$$

and also n and r must satisfy the conditional inequalities:

(a) $r = 0,\ n = p;$
(b) $0 \leq r \leq \frac{1}{3}p,\ 2r \leq n \leq \frac{1}{2}(p + r);$
(c₁) $0 \leq r \leq \frac{1}{2}p,\ 0 \leq n \leq \frac{1}{2}r;$
(c₂) $\frac{1}{2}(p + 1) \leq r < \frac{2}{3}p,\ 2r - p < n \leq \frac{1}{2}r;$
(d) $r = p,\ n = 0.$

4. A numerical example. Theorem 2 makes possible the selection with certainty of the largest possible number of terms of a polynomial of degree p which is termwise invariant under the collineation T. For example, by the laborious process of writing out all terms of a given degree and testing them individually for invariance it is a known result **(1)** that the largest such polynomial for degree 5 is

$$\begin{aligned} F_5 = {} & x_1^5 + x_2^5 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4 + x_1^3x_3x_4 \\ & + x_3^5 + x_2x_3^3x_4 + x_2^2x_3x_4^2 + x_1x_3^2x_4^2 + x_1x_2x_4^3 + x_4^5. \end{aligned}$$

Application of Theorem 2 leads to 12 pairs of values for r and n as follows:

(a) $r = 0, \quad n = 5;$
(b) $r = 0, \quad n = 0, 1, 2;$
$r = 1, \quad n = 2, 3;$
(c) $r = 0, \quad n = 0;$
$r = 1, \quad n = 0;$
$r = 2, \quad n = 0, 1;$
$r = 3, \quad n = 1;$
(d) $r = 5, \quad n = 0.$

The application of these pairs leads precisely to the polynomial F_5.

5. The number of invariant terms. In conclusion we establish

THEOREM 3. *For a prime $p > 3$ the number of invariant terms is equal to* $\frac{1}{6}(p^2 + 6p + 17)$.

Proof. It must first be indicated that all pairs (n, r) selected by Theorem 2 lead to different polynomial terms. We can exclude from further consideration the pair $(0, 0)$ since its appearance in Theorem 2 (b and c_1) produces the polynomials x_2^p, and x_3^p. An examination of the controlling inequalities indicates that all other pairs (n, r) are distinct since (b) and (c_1) differ in n, (b) and (c_2) differ in r, as do (c_1) and (c_2). Then since n is the exponent of x_1 and r is the exponent of x_4, the distinct pairs (n, r) produce distinct polynomial terms.

To count the number of terms in (b) (Theorem 2) for $p = 6\alpha + 1$ we enumerate the pairs (K, L) where $0 \leqq K \leqq \frac{1}{3}(p - 1)$ and $0 \leqq L \leqq \frac{1}{2}(3K + 1)$ as in the accompanying table.

K	0	1	2	3	...	$\frac{1}{3}(p-1)$
L	0	0, 1, 2	0, 1, 2, 3	0, 1, 2, 3, 4, 5	...	$0, \ldots, \frac{1}{2}(p-1)$

The number of terms is thus

$$\begin{aligned}&[1 + 3 + 4 + 6 + 7 + \ldots + \tfrac{1}{2}(p - 1) + \tfrac{1}{2}(p + 1)] \\ &\qquad = [1 + 4 + 7 + \ldots + \tfrac{1}{2}(p + 1)] + [3 + 6 + 9 + \ldots + \tfrac{1}{2}(p - 1)].\end{aligned}$$

By the ordinary formulas for an arithmetic series the number of terms is equal to $(p^2 + 6p + 5)/12$.

By a similar analysis, the inequality (c_1) contributes $(p^2 + 6p + 5)/16$ terms if α is odd, and $(p^2 + 6p + 9)/16$ terms if α is even. In a like manner (c_2) contributes $(p^2 + 6p + 5)/48$ if α is odd and $(p^2 + 6p - 7)/48$ terms if α is even. In either case the number of terms contributed by both (c_1) and (c_2) is $(p^2 + 6p + 5)/12$ since

$$\begin{aligned}(p^2 + 6p + 5)/16 + (p^2 + 6p + 5)/48 &= (p^2 + 6p + 5)/12, \\ (p^2 + 6p + 9)/16 + (p^2 + 6p - 7)/48 &= (p^2 + 6p + 5)/12.\end{aligned}$$

The total number of terms thus obtained which are invariant under T (when $p = 6\alpha + 1$) is thus shown to be $\frac{1}{6}(p^2 + 6p + 17)$. A similar analysis for $p = 6\alpha - 1$ can be shown to yield the same result.

REFERENCES

1. J. Dessart, *Sur les surfaces représentant l'involution engendrée par un homographie de periode cinq du plan*, Mem. Soc. royale des sciences de Liège (3), *17* (1931), 1–23.
2. W. R. Hutcherson, *A cyclic involution of order seven*, Bull. Amer. Math. Soc., *40* (1934), 143–151.

Alabama Polytechnic Institute

A THEOREM ON FIXED POINTS OF INVOLUTIONS IN S^3

DEANE MONTGOMERY AND HANS SAMELSON

1. Introduction. Let T be an orientation-preserving homeomorphism of period two of the 3-sphere S^3 onto itself; further let T be different from the identity and have at least one fixed point. It has been shown by Smith (**8**, p. 162) that the set F of all fixed points of T is a simple closed curve. However, very little is known about the position of F in S^3. There is an example (**4**), a slight variation of an example given by Bing, which shows that F may be wildly imbedded. Even if we assume F to be tame, as we shall do in this paper, it is not known whether T is equivalent to an orthogonal transformation, or even whether F is necessarily unknotted. Our purpose is to show, assuming T semi-linear, that at any rate F cannot belong to a certain class of knots including the ordinary cloverleaf.

We consider S^3 as the unit sphere

$$\sum_1^4 x_i^2 = 1$$

in Euclidean 4-space E^4. "Polyhedral" will always be understood in the sense of spherical geometry; all simplicial decompositions used are obtained by dividing S^3 into simplices by means of planes through the origin of E^4, that is, into spherical simplices; this is equivalent to the decompositions considered in (**5**), where S^3 is considered as boundary of a Euclidean 4-simplex. The orientation-preserving homeomorphism T of period two is assumed to be semi-linear, which by definition means that T is (spherically-) affine on the simplices of a certain simplicial decomposition Σ of S^3; because of the periodicity of T we may assume that in addition T leaves the decomposition Σ invariant.

The set F of fixed points of T is then a simple closed (spherical) polygon (**8**); by going to a subdivision, if necessary, F may be assumed to be a subcomplex of Σ, that is, F is made up of one-cells of Σ.

Let C_1 and C_2 be two simple closed polygons in S^3, disjoint from each other; C_1 is called a parallel knot, of order 2, of C_2, if there exists a polyhedral Möbius band M in S^3 such that C_1 is the boundary of M, and such that C_2 is what we shall call a middle line of M, i.e., such that C_2 represents a generator of the fundamental group $\pi_1(M)$ of M. (By Möbius band we mean the customary figure obtained by identifying two opposite sides of a rectangle, with a twist.) The theorem we propose to prove is as follows:

Received May 3, 1954. The work of the second author was supported by a grant from the National Science Foundation.

THEOREM I. *Suppose the fixed point curve F of the semi-linear involution T of S^3 is a parallel knot, of order 2, of the simple closed polygon C. Then the knot group of C, i.e., the fundamental group $\pi_1(S^3 - C)$ of the complement of C, is infinite cyclic, and the linking number of F and C is ± 1.*

A standard torus W in S^3 is the figure obtained by first constructing a standard torus in E^3 (a circle D in a plane $E^2 \subset E^3$ is revolved around a line $E^1 \subset E^2$ which does not meet D) and then projecting E^3 stereographically onto S^3. The surface W is parametrized by using an angular coordinate r_1 on D and an angular coordinate r_2 for the rotation around E^1. If p, q are relatively prime natural numbers then the curve in S^3 given by

$$r_1 = 2\pi pt, \quad r_2 = 2\pi qt, \qquad 0 \leqslant t \leqslant 1$$

is called the standard torus knot (p, q). A simplicial standard torus knot (p, q) is defined to be a sufficiently close polygonal approximation (obtained by selecting values $0 = t_0 < t_1 < \ldots < t_n = 1$ with max $|t_i - t_{i-1}|$ sufficiently small in an obvious sense and replacing the segment corresponding to $[t_{i-1}, t_i]$ of the original torus knot by the spherical segment connecting the end points).

COROLLARY. *If the fixed point set F of T is a simplicial standard torus knot $(p, 2)$, then $p = 1$, which implies that F is not knotted.*

This follows from Theorem I since the torus knot $(p, 2)$ is a parallel knot, of order 2, of the center line of the torus. It is necessary to observe that all requirements are satisfied simplicially but this is not very difficult to show. Note that the torus knot $(3, 2)$ is the cloverleaf.

The method of proof of Theorem I has as a by-product the following Theorem which (we have learned) has also been proved by E. E. Moise:

THEOREM II. *If F is unknotted in the sense of bounding a polyhedral 2-cell, then T is equivalent to a rotation of S^3.*

In the proofs we make use of a theorem of Alexander **(1)** which states that a polyhedral 2-sphere S^2 in S^3 separates S^3 into two domains A, A' such that $A \cup S^2$ and $A' \cup S^2$ are closed 3-cells (each with boundary S^2). We also use a second theorem by Alexander **(1)** which states that a polyhedral torus B in S^3 separates S^3 into two domains D, D' such that at least one of $B \cup D$, $B \cup D'$ is a solid torus, that is, homeomorphic with the product of a closed disk and a circumference, with B of course corresponding to the boundary. Actually Alexander's proof needs a small modification to become applicable to the spherical polyhedra used here. However Schubert **(6)** has given a proof which is directly applicable to our case.

2. First simplification of $M \cap T(M)$. We now begin the proof of Theorem I and recall that F is boundary of a Möbius band M with middle line C. We can assume that M and C are subcomplexes of the T-invariant simplicial decomposition Σ of S^3. Then $T(M)$ is another simplicial Möbius band and

$T(M)$ also has F as boundary, since F is pointwise fixed. The set $M \cap T(M)$ includes F but must include further points, for otherwise $M \cup T(M)$ would be a Klein bottle imbedded in S^3 without singularity and this is impossible.

We shall show how M may be modified so that $M \cap T(M)$ is made simpler. In the process M remains polyhedral, F remains the boundary, and C is not changed essentially. More specifically, all the Möbius bands considered in the modification process will satisfy the following condition.

Condition (m): *There exists a semi-linear homeomorphism of S^3, which sends some middle line of the band into the curve C (of Theorem I) and leaves F pointwise fixed.* (It is immaterial which middle line is selected because if C_1 and C_2 are two polygonal middle lines of M there is a semi-linear homeomorphism of S^3 onto itself which maps M onto itself, leaves F pointwise fixed, and takes C_1 to C_2. This can be shown by methods indicated in **(6)** and in §4. This fact is implicitly involved in a number of our arguments.)

In the process, subdivisions of Σ may be needed; they will be introduced without explicit mention. The ultimate aim of the modifications is to prove the following:

Lemma A. *Under the hypothesis of Theorem I, there exists a simplicial Möbius band M_1 with boundary F such that*

$$M_1 \cap T(M_1) = F \cup C_1$$

where C_1 is a middle line of M_1 and where condition (m) *is satisfied for C_1.*

The proof is in several parts. The remainder of this section and the next section are devoted to selecting M so that M and $T(M)$ are in general position.

Lemma 1. *Let V be any neighborhood of F. Then there exists a simplicial Möbius band M' such that:*

(1) *the boundary of M' is F;*

(2) *$M \cap (S^3 - V) = M' \cap (S^3 - V)$, that is, M and M' coincide outside of V;*

(3) *there is a neighborhood V' of F such that*

$$M' \cap T(M') - F \subset S^3 - V',$$

that is, the intersection, except for F, is outside of V'. Note that in consequence of (2) *and with V small, a middle line of M is also one of M', so that condition* (m) *is satisfied.*

Proof. By replacing Σ, if necessary, by a subdivision we can assume the following:

(a) F is a normal subcomplex, meaning that every simplex of Σ, whose vertices belong to F, itself belongs to F;

(*b*) the barycentric star St (F) is contained inside V;

(*c*) St(F) is a simplicially divided solid torus whose boundary is a torus surface Q.

As a matter of fact, $\mathrm{St}(F)$ is the union of the 3-cells, corresponding to the vertices of Σ on F in the dual subdivision Σ^* of S^3. Two such 3-cells, corresponding to neighboring vertices of F, have in common a 2-cell, the cell dual to the 1-cell of Σ joining the two vertices, whereas two such 3-cells that correspond to non-neighboring vertices on F are disjoint. We may also assume that M is a normal subcomplex of Σ. It is true that $M \cap \mathrm{St}(F)$ is the barycentric star of F on M. It follows that $M \cap Q$ is a simple closed curve which we shall call D, and that D and F form the boundary of the annulus $M \cap \mathrm{St}(F)$; an annulus is the topological product of an interval I and a circle S^1. As a consequence D is not ~ 0 on Q.

We now propose to construct a polygonal simple closed curve D' on Q such that

(1) $D' \sim D$ on Q

(2) $T(D') \cap D' = 0$ (0 = empty set)

(3) D' and F bound an annulus A in $\mathrm{St}(F)$ for which $A \cap T(A) = F$ and $A \cap Q = D'$. For the construction of D' we note that for each of the 3-cells σ_3, dual to the vertices of F, the boundary 2-sphere $\mathrm{Bd}\,\sigma_3$ intersects Q in an annulus. From the known behavior of periodic transformations of 2-cell and 2-sphere (see **(2)** and the references therein to Brouwer and Kérekjárto) it follows that T on such an annulus $\mathrm{Bd}\,\sigma_3 \cap Q$ is equivalent to the standard transformation of $I \times S^1$, obtained by rotating S^1 through 180°. It is now easy to construct D' as union of polygonal arcs, one arc in each annulus $\mathrm{Bd}\,\sigma_3 \cap Q$, connecting the two components of the boundary of the annulus. The existence of A follows then from the fact that each σ_3 is the join of $\mathrm{Bd}\,\sigma_3$ and the vertex of F, to which it is dual. We now apply Theorem 3, p. 180 (see also p. 161), of Schubert **(6)**; according to this there exists a semi-linear homeomorphism ϕ of S^3 which is the identity outside of V, maps Q into itself and sends D into D'.

We now obtain the desired Möbius band M' as the union of A and the image under ϕ of the part of M in $S^3 - \mathrm{St}(F)$. The requirements of Lemma 1 are now satisfied with V' the interior of $\mathrm{St}(F)$. This completes the proof.

We now call the M' thus obtained again M and proceed with further modifications leading towards proving Lemma A.

3. Reduction to a finite set of simple closed curves. Suppose that $M \cap T(M)$ contains a 2-cell σ of Σ with vertices a, b, c; note that σ is disjoint from F by Lemma 1. Then σ and $T(\sigma)$ have no point in common since otherwise there would be a fixed point of T in $\sigma \cup T(\sigma)$. Now σ is a face of a 3-cell τ of Σ with vertices a, b, c, d. Let e be a point inside τ and subdivide τ into the join $e \circ \mathrm{Bd}\,\tau$ of e and the boundary $\mathrm{Bd}\,\tau$ of τ. Because of the normality of M the intersection $\tau \cap F$ is empty; it follows again that τ and $T(\tau)$ are disjoint since otherwise there would be a fixed point of T in $\tau \cup T(\tau)$. We subdivide $T(\tau)$ by the join $T(e) \circ T(\mathrm{Bd}\,\tau)$; the new subdivision of S^3 is T-invariant.

We now replace the cell σ of M by the join of e and $\mathrm{Bd}\,\sigma$; similarly replace

$T(\sigma)$ in $T(M)$ by the join of $T(e)$ and $T(\mathrm{Bd}\,\sigma)$. The band M' so obtained has boundary F and any alteration of C has been in accord with condition (m). However

$$M' \cap T(M') = M \cap T(M) - [\text{interior } \sigma \cup \text{interior } T(\sigma)];$$

i.e., the interiors of the 2-cells σ and $T(\sigma)$ no longer belong to the intersection of M and $T(M)$, whereas no other intersections are introduced. By a repetition of this process *we arrive at a Möbius band M' such that $M' \cap T(M')$ contains no 2-cell.*

Given a 2-cell σ, as in the above, we are free to make the indicated modification on either side of σ. Further if we have in the intersection a union of 2-cells σ_i of Σ which forms a 2-cell K or an annulus R such that

$$T(K) \cap K = 0 \text{ resp. } T(R) \cap R = 0,$$

then K or R has two well-defined "sides" and we may modify each σ_i to either of the two sides.

Next suppose that $M \cap T(M)$ contains a 1-cell $\eta = (a, b)$ of Σ not on F, and that M and $T(M)$ do not cross each other at η in an obvious sense. In M the 1-cell η lies on two 2-cells, say (a, b, c) and (a, b, d). There exists a set of vertices $d_0 = c, d_1, \ldots, d_k = d$, $1 < k$ such (abd_0d_1), $(abd_1d_2), \ldots (abd_{k-1}d)$ are 3-cells of σ and such that d_i, $0 < i < k$ does not belong to $T(M)$. Using a subdivision of these cells and their images under T we may modify M to become M' in accordance with condition (m) and so that

$$M' \cap T(M') = M \cap T(M) - [\text{interior } \eta \cup \text{interior } T(\eta)].$$

We omit the details of this process of pulling M and $T(M)$ apart. Using this method we obtain a band, again called M, such that M and $T(M)$ cross along any 1-cell in $M \cap T(M) - F$.

Similar methods make it possible to obtain a Möbius band, again called M such that $M \cap T(M)$ contains no isolated points (and of course no 2-cells, and no 1-cells along which M and $T(M)$ do not cross). We note in particular: Any component of the intersection of M and $T(M)$ that is a 2-cell, disjoint from its image, can be completely removed from the intersection (by modifying M in accordance with condition (m)) without introducing new intersections. Any component of $M \cap T(M)$ that is an annulus, disjoint from its image and along which M and $T(M)$ do not cross in the obvious sense, can also be removed in this way. If M and $T(M)$ do cross along such an annulus, then M can be so modified that in the intersection with $T(M)$ there appears instead of the annulus any preassigned simple closed polygon that represents a generator of the fundamental group of the annulus, e.g., either one of the boundary curves; the modified M and $T(M)$ cross each other along this curve. Any component of the intersection that is a simple closed curve along which M and $T(M)$ do not cross can be removed.

Finally let p be a point of M, necessarily a vertex of Σ, such that the order

of the graph $M \cap T(M)$ at p is greater than 2, that is, that more than two 1-cells of $M \cap T(M)$ end at p. Let R be the barycentric star of p, and let E_1 and E_2 be the intersections of M and of $T(M)$ with the boundary 2-sphere Bd R of R. Clearly E_1 and E_2 are simple closed polygonal curves which cross each other at their points of intersection.

The curve E_1 separates Bd R into two 2-cells; let K be one of these. If we replace $R \cap M$ by K and make the corresponding replacement at $T(p)$ on $T(M)$, then on the modified M the number of points of $M \cap T(M)$ of order greater than two has been reduced whereas all other properties are retained. We state the main result of this section as a lemma.

LEMMA 2. *The original Möbius band can be modified to a new band, again called M, satisfying condition* (m) *and such that the intersection $M \cap T(M)$ consists of a finite number of simple closed curves, along which M and $T(M)$ cross each other.*

4. Curves on a Möbius band. For further simplifications we shall need information about the simple closed curves on a Möbius band, which we formulate as a lemma.

LEMMA 3. *Let $\tilde{M}$ be any Möbius band. Any simple closed curve $\tilde{C}$ on $\tilde{M}$ belongs to exactly one of the following three, topologically invariant, categories:*

(1) *the curves which are contractible to a point;*

(2) *the curves which represent a generator of the fundamental group $\pi_1(\tilde{M})$ of $\tilde{M}$; we call these middle lines of $\tilde{M}$;*

(3) *the curves which represent the square of a generator of $\pi_1(\tilde{M})$; we call these edge-like (the boundary of $\tilde{M}$ is in this category).*

A middle line, disjoint from the boundary of $\tilde{M}$, has arbitrarily small neighborhoods on $\tilde{M}$ that are Möbius bands. An edge-like curve, disjoint from the boundary of $\tilde{M}$, separates $\tilde{M}$ into a Möbius band and an annulus; it has arbitrarily small neighborhoods on $\tilde{M}$ homeomorphic to an annulus.

Any two middle lines intersect. Any two disjoint edge-like curves bound an annulus contained in $\tilde{M}$; a middle line, disjoint from the two edge-like curves, does not meet the annulus bounded by them.

We omit the proofs of these statements and only note the following: The Möbius band has as an orientable double covering an annulus R, represented by the region $1 < x^2 + y^2 < 2$ in an (x, y)-plane; the Möbius band is obtained by considering the involutory transformation α of R onto itself, given, in polar coordinates, by

$$\alpha(r, \theta) = (3 - r, \theta + \pi),$$

and identifying pairs of points, which correspond under α, to single points. The identification amounts to a map H from R to the Möbius band, which is the covering map. Proofs for the statements above on curves on $\tilde{M}$ can be made by considering the inverse images under H.

5. Elimination of bounding curves. Using the facts of §4 on curves on a Möbius band we can now make further simplifications.

LEMMA 4. *The band M can be modified (consistent with condition* (m)) *so that no component of $M \cap T(M)$ bounds a 2-cell on M or on $T(M)$.*

For the proof we start with the Möbius band M obtained in Lemma 2. Let $C_1, \ldots, C_n$ be those simple closed curves in $M \cap T(M)$, which bound 2-cells in M. We can assume that C_1 is such that the 2-cell K_1, bounded by it on M, contains no point of $M \cap T(M)$ in its interior. We show first that C_1 also bounds a 2-cell on $T(M)$. If not, then it is edge-like or a middle line (in the sense of §4) on $T(M)$.

If it is edge-like on $T(M)$, then it bounds a Möbius band M' on $T(M)$. The set $M' \cup K_1$ would then be a projective plane imbedded in S^3, which is well known to be impossible.

If it is a middle line, then one gets a contradiction by considering the intersection curves of K_1 and $T(M)$ with the boundary torus of the barycentric star of C_1.

We now show how to remove C_1 from $M \cap T(M)$, without introducing new intersections (and of course still satisfying condition (m)). We show first that C_1 cannot meet $T(C_1)$. If it did, then the two would be identical, since T permutes the components of $M \cap T(M)$. Then $K_1 \cup T(K_1)$ would be a 2-sphere, invariant under T. Since this 2-sphere does not meet F, the two 3-cells, into which S^3 is divided by it according to Alexander's theorem **(1)**, would have to be invariant, and would therefore contain fixed points of T in their interiors. But this contradicts the fact that F is connected. We have therefore $C_1 \cap T(C_1) = 0$. As shown before, C_1 bounds a 2-cell K'_1 in $T(M)$. We consider now two cases,

(I): $K_1 \cap T(K'_1) = 0$,

(II): $K_1 \cap T(K'_1) \neq 0$.

We start with case (I).

As a preliminary modification we define a set M^* by

5.1 $$M^* = (M - T(K'_1)) \cup T(K_1).$$

We have of course

5.11 $$T(M^*) = (T(M) - K'_1) \cup K_1.$$

M^* is again a Möbius band, since we have replaced the cell $T(K'_1)$ by the cell $T(K_1)$, which has no point of M in its interior; note that $T(K_1)$ and $T(K'_1)$ have the same boundary curve. Condition (m) concerning the middle line is still satisfied. The intersection of M^* and its transform is given by

5.2 $$\begin{aligned} M^* \cap T(M^*) &= ((M - T(K'_1)) \cap (T(M) - K'_1)) \cup ((M - T(K'_1) \cap K_1) \\ &\qquad \cup ((T(M) - K'_1) \cap T(K_1)) \cup (K_1 \cap T(K_1)). \end{aligned}$$

Since $K_1 \cap T(K'_1) = 0$, the second term on the right is just K_1; similarly the third term is $T(K_1)$. As a consequence, we have

5.3 $$K_1 \cup T(K_1) \subset M^* \cap T(M^*) \subset (M \cap T(M)) \cup K_1 \cup T(K_1).$$

The two 2-cells K_1 and $T(K_1)$ are disjoint; this follows from the fact that C_1 and $T(C_1)$ are disjoint, and that the interior of K_1 does not meet $T(M)$ at all. By the method described in §3, we can therefore remove K_1 and $T(K_1)$ from the intersection of M^* and $T(M^*)$ without introducing new intersections, and arrive at a new Möbius band M_1, such that

$$M_1 \cap T(M_1) \subset M \cap T(M) - (C_1 \cup T(C_1)),$$

i.e., the intersection of M_1 and $T(M_1)$ is contained in that of M and $T(M)$, but does not contain the curves C_1 and $T(C_1)$ any more; this finishes case (I).

In case (II) we define M^*, and $T(M^*)$ as in case (I) by 5.1 and 5.11. For the intersection $M^* \cap T(M^*)$ we have again formula 5.2. It is again true that $K_1 \cap T(K_1) = 0$. Further we now actually have $K_1 \subset T(K'_1)$, since the boundary curve $T(C_1)$ of $T(K'_1)$ is disjoint from C_1, as shown above, and therefore also from K_1. Similarly $T(K_1) \subset K'_1$. It follows then from 5.2 that

5.4 $$M^* \cap T(M^*) = (M - T(K'_1)) \cap (T(M) - K'_1),$$

and so

5.5 $$M^* \cap T(M^*) \subset M \cap T(M) - (C_1 \cup T(C_1))$$

(note that C_1 is the boundary curve of K'_1, and $T(C_1)$ that of $T(K'_1)$), so that C_1 and $T(C_1)$ are no longer in the intersection. This finishes case (II); condition (m) is still satisfied.

Iteration of this process must come to an end, since at each application the number of components of $M \cap T(M)$ decreases, and Lemma 4 follows.

6. Reduction to a middle line. We now take M to satisfy the condition of Lemma 4, and proceed with a last reduction, in order to arrive at Lemma A. The intersection $M \cap T(M)$ now consists of F and a number of curves, say $C_1, \ldots, C_m$, which are either edge-like or middle lines on M, with at most one in the last category (because of Lemma 3). Suppose there are actually edge-like curves in this collection. One concludes, with the help of Lemma 3, that there is one of these, which we can assume to be C_1, such that the annulus R bounded on M by F and C_1 contains no point of $M \cap T(M)$ in its interior. We show that C_1 cannot be invariant under T, or, which amounts to the same, that $C_1 \cap T(C_1) = 0$. If $C_1 = T(C_1)$, then $R \cup T(R)$ would be a T-invariant torus, which separates S^3 into two domains. From the action of T near F it follows that T interchanges the two domains. On the other hand, from the action of T near C_1 we see that T cannot interchange the two domains. This contradiction establishes our assertion.

The curve $T(C_1)$ is therefore one of $C_2, \ldots, C_m$; we claim that it is edge-like on M. If it were a middle line, we would get a contradiction by considering

the intersection curves of M and $T(M)$ with the boundary of its barycentric neighborhood, as in the reasoning of §5; recall that $T(C_1)$ is edge-like on $T(M)$. In particular, the number of edge-like curves among $C_1, \ldots, C_m$ is >1.

With the help of Lemma 3 we see that there is a curve in $M \cap T(M)$, edge-like on M and on $T(M)$, which we can call C_2, such that the interior of the annulus R_1 bounded by C_1 and C_2 on M does not meet $T(M)$. Let R'_1 be the annulus bounded by C_1 and C_2 on $T(M)$; the interior of R'_1 may meet M. We have two possibilities:

(A) $$C_2 = T(C_1),$$

(B) $$C_2 \neq T(C_1).$$

In case (A) we have $C_1 = T(C_2), R'_1 = T(R_1), R_1 = T(R'_1)$. We define then

$$M^* = (M - R_1) \cup T(R_1),$$

so that

$$T(M^*) = (T(M) - T(R_1)) \cup R_1.$$

M^* is again a Möbius band. The intersection $M^* \cap T(M^*)$ is identical with $M \cap T(M)$; but M^* and $T(M^*)$ do not cross along C_1 and C_2, so that C_1 and C_2 can be removed from the intersection by the methods of §4; condition (m) is kept intact.

In case (B) we have to distinguish two subcases:

(I) $R_1 \cap T(R'_1) = 0$,

(II) $R_1 \cap T(R'_1) \neq 0$.

We begin with (I). Then R_1 and $T(R'_1)$ are non-intersecting annuli on M. We define new sets by

6.1 $$M^* = (M - T(R'_1) \cup T(R_1),$$

6.11 $$T(M^*) = (T(M) - R'_1) \cup R_1.$$

Then M^* is again a Möbius band; an annulus between two edge-like curves has been replaced by an annulus; this has no effect on the middle lines, so that condition (m) is satisfied. For the intersection with $T(M^*)$ we have the formula

6.2 $$\begin{aligned} M^* \cap T(M^*) &= (M - T(R'_1)) \cap (T(M) - R'_1) \cup (M - T(R'_1)) \cap R_1 \\ &\quad \cup (T(M) - R'_1) \cap T(R_1) \cup R_1 \cap T(R_1). \end{aligned}$$

Because of (I) the second and third term are just R_1 and $T(R_1)$. It follows that

6.3 $$M^* \cap T(M^*) = (M - T(R'_1)) \cap (T(M) - R'_1) \cup R_1 \cup T(R_1).$$

As in §5 we see that R_1 and $T(R_1)$ are disjoint: the T-image of the boundary of R_1 does not meet R_1, since we have case (I), and the interior of R_1 does not meet $T(M)$ at all. We are therefore in the situation of §3: R_1 and $T(R_1)$ are

disjoint components of $M^* \cap T(M^*)$. We can therefore replace M^* by another permissible Möbius band such that either R_1 and $T(R_1)$ are absent from the intersection with the T-transform (this if M^* and $T(M^*)$ do not cross along R_1) or R_1 is replaced by C_2, and $T(R_1)$ by $T(C_2)$ (this if M^* and $T(M^*)$ do cross along R_1). In either case C_1 and $T(C_1)$ have been removed from the intersection.

We come to case (II). $T(R'_1)$ is an annulus on M, bounded by $T(C_1)$ and $T(C_2)$. Because of our assumptions $T(C_1)$ is different from C_1 and C_2. We show first that we cannot have $C_2 = T(C_2)$. We assume then for the moment that C_2 is invariant. Let R_2 be the annulus bounded by F and C_2 on M; its boundary is then invariant. If the interiors of R_2 and $T(R_2)$ meet, this can happen only along C_1, so that then C_1 would be T-invariant; but this we have shown to be impossible. It follows that $R_2 \cup T(R_2)$ is a T-invariant torus. The reasoning applied to $R \cup T(R)$ in the beginning of this section applies here too, and shows that C_2 is different from $T(C_2)$. It follows that the boundary of $T(R'_1)$ is disjoint from R_1. Since R_1 is connected, it would have to be contained in the interior of $T(R'_1)$. But this is impossible, since no component of $M \cap T(M)$ lies in the annulus R between F and C_1 on M, and so case (II) cannot occur at all.

The processes described in this section can be applied as long as there are edge-like curves, different from F, in the intersection of the band and its transform. By a finite number of steps we arrive therefore at a Möbius band M_1 satisfying Lemma A; as noted earlier, the set $M_1 \cap T(M_1) - F$ cannot be empty, and on the other hand, two middle lines on a Möbius band are never disjoint, by Lemma 3, so that exactly one middle line will be left over in the intersection (besides F).

7. The proof of Theorem I. The knot groups of the original curve C and of the curve C_1 of Lemma A are isomorphic, and the linking numbers of F with C and C_1 are equal, because of condition (m). It is therefore sufficient to prove Theorem I for C_1 instead of C. We return to the notation M for the Möbius band, obtained in Lemma A, and C for the single (invariant) curve constituting $M \cap T(M) - F$. We denote the polyhedron $M \cup T(M)$ by S; it is clear that $T(S) = S$. The second homology group $H_2(S)$ is infinite cyclic and the generating 2-cycle contains all 2-cells, properly oriented, with coefficient $+1$. (In fact S can be described as obtained from a torus $W_2 = S^1 \times S^1$ by identifying pairs of antipodal points on some generating circle $p \times S^1$.)

It follows from Alexander's duality theorem that $S^3 - S$ has two components which we call A_1 and A_2. We have $A_2 = T(A_1)$, $A_1 = T(A_2)$. The reason is that T, at any point of F, at the same time reverses the orientation of S and preserves the orientation of the 3-sphere; it interchanges therefore the two domains into which S separates the 3-sphere locally (and globally). It is also true that $\mathrm{Cl}A_i$, the closure of A_i, is $A_i \cup S$, and that $\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2 = S$.

Let N be the closed barycentric neighborhood of the middle curve C with

respect to the triangulation of S^3 reached after the various modifications. Then N is a solid torus and its boundary $\mathrm{Bd}N = B$ is a torus.

One shows with elementary deformations that the three sets $S^3 - N$, the closure of $S^3 - N$, and $S^3 - C$ have isomorphic fundamental groups, and that in fact the inclusion maps induce these isomorphisms. The intersection

$$P = N \cap M$$

is the closed barycentric neighborhood of C on M. It is a Möbius band with C as middle line; similar remarks apply to

$$Q = N \cap T(M).$$

It can be shown by elementary constructions that the following facts hold:

The set $N - (P \cup Q)$ consists of two connected sets N_1, N_2 which are the intersections of N with A_1, A_2.

The boundary curves $\mathrm{Bd}P$, $\mathrm{Bd}Q$ of P and Q, which lie on B, separate B into sets B_1, B_2 each homeomorphic with an open annulus, and $B_i = B \cap A_i$; the closure $\mathrm{Cl}B_i$ of B_i is $B_i \cup \mathrm{Bd}P \cup \mathrm{Bd}Q$. The generators of the fundamental group of $\mathrm{Cl}B_i$ represented by $\mathrm{Bd}P$ or $\mathrm{Bd}Q$ are homotopic in N to the square of the generator of the fundamental group of N which is in turn represented by C. All sets N, $\mathrm{Cl}N_i$, P, Q are deformation retractable onto C; also $\mathrm{Cl}A_i$ is a deformation retract of $\mathrm{Cl}A_i \cup N$. Note that N and B are invariant under T and that P and Q, $\mathrm{Bd}P$ and $\mathrm{Bd}Q$, N_1 and N_2, B_1 and B_2 are interchanged by T.

Finally let

$$X = \mathrm{Cl}(S - (P \cup Q)).$$

This is an annulus for which F, the fixed point curve of T, is a generator of the fundamental group. We shall now prove the first half of Theorem I.

PROPOSITION 1. *The fundamental group of* $S^3 - C$ *or of* $\mathrm{Cl}(S^3 - N)$ *is infinite cyclic.*

Proof. We form $X \cup B_1$, which by construction is a torus; it is also the boundary of each of the polyhedra $N \cup \mathrm{Cl}A_2$ and $\mathrm{Cl}(A_1 - N)$. By Alexander's theorem **(1)** one of these two is homeomorphic with a solid torus. We consider two cases.

(a) Suppose that $\mathrm{Cl}(A_1 - N)$ is a solid torus. Then $\mathrm{Cl}(A_2 - N)$ is also a solid torus, as T-image of $\mathrm{Cl}(A_1 - N)$. The union of $\mathrm{Cl}(A_1 - N)$ and $\mathrm{Cl}(A_2 - N)$ is $\mathrm{Cl}(S^3 - N)$; the intersection of $\mathrm{Cl}(A_1 - N)$ and $\mathrm{Cl}(A_2 - N)$ is X. To compute $\pi_1 \mathrm{Cl}(S^3 - N)$, we use the addition theorem **(7**, p. 177**)**, according to which $\pi_1 \mathrm{Cl}(S^3 - N)$ is the free product of $\pi_1 \mathrm{Cl}(A_1 - N)$ and $\pi_1 \mathrm{Cl}(A_2 - N)$ with the additional relations obtained by equating elements which correspond to the same element of $\pi_1(X)$. Let g_1 and g_2 be the generators of the (infinite cyclic) groups $\pi_1 \mathrm{Cl}(A_1 - N)$ and $\pi_1 \mathrm{Cl}(A_2 - N)$; F, considered as a curve in $\mathrm{Cl}(A_1 - N)$, represents some power g_1^m of g_1. Since $T(F) = F$ and $T(g_1) = g_2$, there will be only one new relation, namely, $g_1^m = g_2^m$. The

homology group $H_1\mathrm{Cl}(S^3 - N)$ is then the abelian group with two generators γ_1 and γ_2 and the relation $m\gamma_1 = m\gamma_2$. Because of the Alexander duality theorem this group must be infinite cyclic; it follows that $m = \pm 1$. But this clearly means that $\pi_1\,\mathrm{Cl}(S^3 - N)$ is infinite cyclic, and that F represents a generator.

(b) Suppose that $\mathrm{Cl}A_2 \cup N$ is a solid torus. We show first that the curve C represents a generator of $\pi_1(\mathrm{Cl}A_2 \cup N)$. For the proof we can consider $\mathrm{Cl}A_2$ instead of $\mathrm{Cl}A_2 \cup N$, since it is a deformation retract of the latter. C represents some power β^k of a generator β of $\pi_1\mathrm{Cl}A_2$; we wish to show $k = \pm 1$, and assume temporarily that this is not so. It follows, going to the homology, that C is homologous to 0 mod k in $\mathrm{Cl}A_2$. Applying T, one gets the same behavior in $\mathrm{Cl}A_1$. On the other hand, from the explicit structure of $S = \mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$ we see that C is not homologous to 0 mod k on S. But this leads to a contradiction with the Mayer-Vietoris theorem **(3)** for the decomposition $S^3 = \mathrm{Cl}A_1 \cup \mathrm{Cl}A_2$, since $H_2(S^3) = 0$. It follows that $k = \pm 1$, as we claimed.

We now represent $\mathrm{Cl}A_2 \cup N$ as union of the two polyhedra $\mathrm{Cl}(A_2 - N)$ and N, whose intersection is $\mathrm{Cl}B_2$. Let γ denote the generator of $\pi_1(N)$, represented by C. A generator of $\pi_1(\mathrm{Cl}B_2)$, e.g., $\mathrm{Bd}P$, represents then the element γ^2 in $\pi_1(N)$, and a certain element α in $\pi_1\mathrm{Cl}(A_2 - N)$. The group $\pi_1(\mathrm{Cl}A_2 \cup N)$ is obtained, according to the addition theorem used above, by adding the new generator γ to the generators of $\pi_1\mathrm{Cl}(A_2 - N)$ and adding the relation $\alpha = \gamma^2$ to the relations in $\pi_1\mathrm{Cl}(A_2 - N)$. Since γ is represented by C, it follows that γ is a generator of the group $\pi_1(\mathrm{Cl}A_2 \cup N)$, which is infinite cyclic, since $\mathrm{Cl}A_2 \cup N$ is a solid torus. Consequently the element α of $\pi_1\mathrm{Cl}(A_2 - N)$ is also of infinite order, and we see that $\pi_1(\mathrm{Cl}A_2 \cup N)$ is the free product of $\pi_1\mathrm{Cl}(A_2 - N)$ and $\pi_1(N)$ with amalgamated subgroups $\{\gamma^2\}$ and $\{\alpha\}$ in the sense of Schreier **(5)**. It follows from this theory that $\pi_1\mathrm{Cl}(A_2 - N)$ is isomorphically contained in $\pi_1(\mathrm{Cl}A_2 \cup N)$, and is therefore itself infinite cyclic. But then the argument of (a) can be used to prove Proposition 1.

We now come to the second part of Theorem I:

PROPOSITION 2. *The linking number of F and C is* ± 1.

Proof. The reasoning of (a) and (b) above showed that the groups $\pi_1\mathrm{Cl}(A_i - N)$ and $\pi_1\mathrm{Cl}(S^3 - N)$ are infinite cyclic, with F representing a generator of $\pi_1\mathrm{Cl}(S^3 - N)$ and so also of $\pi_1(S^3 - C)$. Since obviously there are elements in $\pi_1(S^3 - C)$, whose linking number with C is ± 1, it follows that F must also have this property, and Theorem I is proved.

8. The proof of Theorem II. Again T is an involution of S^3 which preserves orientation with fixed point curve F; now F is unknotted. This hypothesis means that F bounds a cell N where N is a polyhedron of an invariant subdivision of S^3. Earlier arguments show:

(1) that N may be modified (§3) so that $N \cap TN$ consists of a finite set of simple closed curves;

(2) that by a further reduction (§5) it can be arranged that

$$N \cap TN = F.$$

Then $N \cup TN$ is a 2-sphere (polyhedral) and S^3 is the union of two closed 3-cells, L and $T(L)$, each having for boundary the 2-sphere $N \cup TN$. Let T_0 be the rotation of S^3, of period 2, given by $x_1 \to -x_1$, $x_2 \to -x_2$, $x_3 \to x_3$, $x_4 \to x_4$. The fixed point curve F_0 is given by $x_1 = x_2 = 0$. An invariant sphere is given by $x_1 = 0$; it is divided by F_0 into two 2-cells N_0 and $T_0(N_0)$, given by $x_1 = 0, x_2 > 0$, respectively $x_2 < 0$. The whole sphere S^3 is the union of the two 3-cells L_0 and $T_0(L_0)$, given by $x_2 > 0$, respectively $x_2 < 0$, each having for boundary the sphere $N_0 \cup T_0(N_0)$.

We begin now by setting up a homeomorphism ϕ between N and N_0, arbitrarily chosen. This can be extended to a homeomorphism ϕ between $N \cup T(N)$ and $N_0 \cup T_0(N_0)$ by defining $\phi = T_0 \cdot \phi \cdot T$ on $T(N)$. The homeomorphism ϕ between $N \cup T(N)$ and $N_0 \cup T_0(N_0)$ can be extended to a homeomorphism ϕ between L and L_0. This in turn can be extended to a homeomorphism of S^3 with itself by defining $\phi = T_0 \cdot \phi \cdot T$ on $T(L)$. The ϕ so constructed satisfies $\phi = T_0 \cdot \phi \cdot T$, or $T = \phi^{-1} \cdot T_0 \cdot \phi$, which shows that our involution T is equivalent to the rotation T_0.

References

1. J. W. Alexander, *On the subdivision of 3-space by a polyhedron*, Proc. Nat. Acad. Sci., *10* (1924), 6–8.
2. S. Eilenberg, *Sur les transformations périodiques de la surface de la sphère*, Fund. Math., *22* (1934) 28–41.
3. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, (Princeton, 1952).
4. D. Montgomery and L. Zippin, *Examples of transformation groups*, Proc. Amer. Math. Soc., to appear.
5. O. Schreier, *Die Untergruppen der freien Gruppen*, Abh. Math. Sem. Hamburg, *5* (1927), 161–183.
6. H. Schubert, *Knoten und Vollringe*, Acta Math., *90* (1953), 132–286.
7. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, (Leipzig, 1934).
8. P. A. Smith, *Transformations of finite period*, Ann. Math., *39* (1938), 127–164.

Institute for Advanced Study

FULL INDIVIDUAL AND CLASS DIFFERENTIATION THEOREMS IN THEIR RELATIONS TO HALO AND VITALI PROPERTIES

C. A. HAYES, Jr. and C. Y. PAUC

Introduction. In his article **(22)**, de Possel laid the foundations for an *abstract theory of differentiation of set functions*, the term "abstract" being meant in the sense of Fréchet-Nikodym, that is, without reference to a euclidean, metric, or topological background. In 1.1, we adopt, substantially, his notion of *derivation basis*. De Possel considered two Vitali properties for a derivation basis. The *strong* or classical Vitali property asserts the existence of an enumerable disjointed p.p. covering family; it implies the *full differentiation* theorem for integrals, that is, the existence almost everywhere of the derivative and its equality with a Radon-Nikodym integrand. The *weak* Vitali property asserts the existence of a p.p. covering family with arbitrarily small *overlap*; it is equivalent to the *density property* or the *full differentiability of Lipschitzian integrals*. One of us, in **(10)** and **(11)**, introduced in Morse's setting of **(14)**, two variations of the weak Vitali property. In the *pseudo-strength* the overlap refers to any Radon measure; the *$\mathfrak{L}^{(p)}$-overlap property* ($p > 1$) involves the pth power of the excess of covering function. The pseudo-strength implies the existence almost everywhere of the derivative of any Radon measure. The $\mathfrak{L}^{(p)}$-overlap property does the same for the integrals of $\mathfrak{L}^{(q)}$-functions ($p^{-1} + q^{-1} = 1$). In §1, these Vitali properties are transferred to a general derivation basis $\bar{B}$. The missing topology is replaced by the pretopology **(21)** which is derived from $\bar{B}$. Individual and class differentiation theorems are established for integrals and Radon measures. The technique at first follows de Possel's trend. Section 2 deals with the following converse problem: Do full differentiation assertions for sigma-additive set functions imply covering properties of Vitali types? It suffices to refer to the proof of the Zygmund theorem in **(29)**, and of the Zygmund-Marcinkiewitz-Jessen theorem in **(25)** concerning the interval basis, to realize that not all differentiation proofs rest on Vitali properties. By an adaptation of de Possel's proof of his equivalence theorem, we show that the full differentiability of integrals or Radon measure is equivalent to a Vitali property (2.2). Here again we proceed from individual to class assumptions. We prove (2.4) that the full differentiability of integrals of $\mathfrak{L}^{(q)}$-functions ($q > 1$) implies the $\mathfrak{L}^{(p')}$-overlap property for $p' < p$, when $\bar{B}$ is a $\mathfrak{D}$-basis. Thus the interval basis possesses the $\mathfrak{L}^{(p)}$-overlap property for any (finite) $p > 1$. In the classical proof by Carathéodory of the Lebesgue differentiation theorem for the cube basis, the preliminary Vitali theorem is deduced from a *halo property* of cubes,

Received January 23, 1954; in revised form August 14, 1954.

namely: If for any cube V_0 (*nucleus*), $H(V_0)$ (*halo*) denotes the union of those cubes V which are not greater than V_0 and intersect V_0, then the *dilatation*, that is, the ratio of the measure of the halo to the measure of V_0, is uniformly bounded for all V_0, in fact equals 3^n, where n denotes the dimension of the euclidean space. Halo properties differ by the requirements for the $\bar{B}$-sets constituting the halo, mainly by the *incidence* requirements; in the example just given, the non-vacuity of $V \cdot V_0$ was demanded. In **(1)**, Busemann and Feller gave, for the special euclidean bases considered by them, a *weak halo property equivalent to the density property*. In 2.5, we give an individual differentiability criterion of Busemann-Feller type, thus shedding light on a second converse problem: Do full differentiation properties for σ-additive set functions imply halo properties? In this connection we mention that a halo property of Busemann-Feller type creeps into the proof of 2.4. Morse, in his fundamental memoir **(14)**, formulated halo conditions, securing the strong Vitali property for his *blankets*. He assumes that the $\bar{B}$-sets are closed; but he also shows that his differentiation theorems remain valid when this assumption is dropped. In §3 we prove that in our setting, the pointwise halo condition implies the Vitali property for integrals or Radon measures (3.2). We give two examples (3.3 and 3.4), where the surrender of the closeness of the $\bar{B}$-sets leads to the substitution in the assertions of the new Vitali property (pseudo-strength) in place of the strong Vitali property. In §4 we tackle the differentiation of functions λ defined on the $\bar{B}$-sets. Our main tool is the *Vitali integration* to transform the "interval" function λ into a set function which is expected to turn out to be an integral ψ on the measurable sets. When this proves true, then the differentiability study of λ reduces to that of ψ and our methods of §1 are applicable. Our results contain as a special case those published by Morse in **(14)** on the differentiation of *addivelous functions*. The authors wish to acknowledge with thanks helpful suggestions made by K. O. Househam in the course of many discussions.

§1. Differentiation of σ-Additive Set Functions under Covering Assumptions of Vitali Type

1.1. Setting. R denotes a set of points, which is our universe. **S** denotes the Boolean σ-algebra[1] of all subsets of R.

For two sets X and Y belonging to **S**, $X \supset Y$ means ordinary inclusion, permitting the equality $X = Y$.

We use both the lattice-theoretical symbols $\cup$, $\cap$, $\bigcup$, $\bigcap$, and the algebraic symbols $+$, $-$, and ., in Stone's sense. However, we generally use the latter only when Stone's and Hausdorff's (set-theoretical) meaning coincide.

M denotes a Boolean σ-algebra of subsets of R with R as unit; μ represents a fixed σ-finite measure defined on **M**; μ^* is the completion in **S** of μ, defined on **M***. Also, $\bar{\mu}$ represents the outer measure derived from μ (or, equivalently,

[1]Definition of Boolean σ-algebras and other related terms may be found in **(4**, pp. 19–26**)**.

from μ^*), defined on **S**, namely $\bar{\mu}(S) = \inf \mu(M)$, where the infimum is taken over all sets M such that $S \subset M$ and $M \in \mathbf{M}$. Similarly, we define $\underline{\mu}$ on **S** by $\underline{\mu}(S) = \sup \mu(M)$, where the supremum is taken over all sets M such that $S \supset M$ and $M \in \mathbf{M}$.

N denotes the family of the μ-nullsets, which is a σ-ideal in **M** (regarded as a Boolean σ-ring); $\mathbf{N}^*$ is the family of the μ^*-nullsets, which is a σ-ideal in **S** (envisaged as a Boolean σ-ring).

By $X \supset Y \pmod{\mathbf{N}}$ we shall mean $Y - X\cdot Y \in \mathbf{N}$; $X = Y \pmod{\mathbf{N}}$ will be understood to mean that Stone's difference $X - Y = [(X - X\cdot Y) + (Y - X\cdot Y)] \in \mathbf{N}$.

For $S \in \mathbf{S}$, a *μ-cover* $\bar{S}$ of S is any **M**-set for which $\bar{S} \supset S$, and $\bar{\mu}(S\cdot M) = \mu(\bar{S}\cdot M)$ for any $M \in \mathbf{M}$. Similarly **(3**, p. 68**)**, a *μ-kernel* $\underline{S}$ of S is any **M**-set such that $\underline{S} \subset S$, and $\underline{\mu}(S\cdot M) = \mu(\underline{S}\cdot M)$ for any $M \in \mathbf{M}$.

Two sets S' and S'' are said to be *μ^*-entangled* if they have positive outer measure and common μ-cover.

We define a *derivation basis* $\bar{\mathcal{B}}$ as follows. We assume that to each point x of a fixed subset E of R, there correspond sequences, in the sense of Moore-Smith, of **M**-sets of finite positive measure, called *constituents*, which are said to *converge* to x, and are denoted generically by $M_\iota(x)$. Further, we assume de Possel's heredity (or Fréchet's convergence) axiom[2]; namely, every (cofinal) subsequence of an x-converging sequence itself converges to x. The family of the sequences $M_\iota(x)$ is our derivation basis $\bar{\mathcal{B}}$. The elements of $\bar{\mathcal{B}}$ are thus converging sequences, together with corresponding convergence points. (This notion involves a basic measure μ. The correspondence of converging sequences to points is called *prebasis* by Haupt and Pauc in **(9)**. A prebasis defines a *pretopology* **(21**, §2**)**. Some pretopological notions involve a σ-ideal of nullsets). The definition just given does not exclude the possibility that two distinct points correspond to the same converging sequence. We denote by **D** the family of sets occurring in the sequences $M_\iota(x)$ for all $x \in E$. If λ is a numerical function defined on the **D**-sets, and $x \in E$, then we define

$$D^* \lambda(x) = \sup\left[\limsup \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right],$$

where the expression in brackets denotes the limit superior for any one x-converging sequence $M_\iota(x)$, and the supremum is taken among all sequences converging to x. In exactly similar fashion we define

$$D_* \lambda(x) = \inf\left[\liminf \frac{\lambda(M_\iota(x))}{\mu(M_\iota(x))}\right].$$

We call $D^*\lambda(x)$ and $D_*\lambda(x)$ the *upper* and *lower $\bar{\mathcal{B}}$-derivates* at x, respectively. If $D^*\lambda(x) = D_*\lambda(x)$ (finite or infinite), we say that the *$\bar{\mathcal{B}}$-derivative* $D\lambda(x) =$

[2]This is introduced in **(22**, p. 397**)**. The limitation to ordinary sequences $i = 1, 2, \ldots$ is irrelevant throughout de Possel's paper.

$D^*\lambda(x) = D_*\lambda(x)$ *exists at* x, or that λ is *$\bar{B}$-differentiable at* x. If the sequences $M_\iota(x)$ are subsequences of one universal sequence **(14)**, then we can drop the prefixes "sup" and "inf" in the expressions for $D^*\lambda(x)$ and $D_*\lambda(x)$.

By *subbasis of* $\bar{B}$, we mean any subfamily $\bar{B}^*$ of $\bar{B}$, containing all subsequences of any of its sequences, retaining the corresponding convergence points. The family of the constituents occurring in the *$\bar{B}^*$-sequences* is called the *spread* of $\bar{B}^*$; the set of points $D(\bar{B}^*)$, each of which is a convergence point of at least one $\bar{B}^*$-sequence, is called the *domain* of $\bar{B}^*$. The spread $\mathbf{V} = \mathbf{V}(X)$ of any subbasis $\bar{B}^*$ with $D(\bar{B}^*) \supset X$ (mod $\mathbf{N}^*$) is called a *$\bar{B}$-fine covering* of X. A $\bar{B}$-fine covering $\mathbf{V} = \mathbf{V}(X)$ of a set X may also be defined as a family of constituents containing, for almost every x (that is, everywhere but on an $\mathbf{N}^*$-set) in X, the sets of at least one sequence $M_\iota(x)$.

The importance of the latter notion for the theory of differentiation results from the following considerations. If $X \subset [D^*\lambda > \alpha]$, then the family of those constituents M satisfying $\lambda(M) > \alpha\mu(M)$ is a $\bar{B}$-fine covering of X. The same is true if "D^*" is replaced by "D_*" and "$>$" by "$<$" in the preceding sentence.

In the second definition of a $\bar{B}$-fine covering, the requirement of the existence of at least one sequence $M_\iota(x)$ may be replaced by the following stronger one: Every x-converging sequence admits of a subsequence, the sets to which belong to $\mathbf{V}$. When this condition holds, we shall say **(21**, p. 74**)** that $\mathbf{V}$ is a *full $\bar{B}$-fine covering of* X. This new requirement is equivalent to the apparently stronger one: For every x-converging sequence S consisting of the sets M_ι there exists an index $\iota' = \iota'(S)$ such that $\iota > \iota'$ implies $M_\iota \in \mathbf{V}$. The intersection of two full $\bar{B}$-fine coverings of X is again a full $\bar{B}$-fine covering of X; the intersection of a $\bar{B}$-fine covering of X and a full $\bar{B}$-fine covering of X is a $\bar{B}$-fine covering of X.

With the same notation as above, the family of those constituents M satisfying $\lambda(M) > \alpha\mu(M)$ is a full $\bar{B}$-fine covering of any set $X \subset [D_*\lambda > \alpha]$. The same is true if "$D_*$" and "$>$" are replaced by "$D^*$" and "$<$", respectively, in the foregoing sentence.

A point x is termed *totally interior* (with respect to $\bar{B}$) to a subset X of R, if, for every x-converging sequence S consisting of the sets M_ι, there exists **(18)** an index $\iota' = \iota'(S, x)$, such that $\iota > \iota'$ implies $M_\iota \subset X$. We represent by $I(X)$ the set of points x which are totally interior to X. $I(X)$ need not be a subset of X. (In the case of a blanket F, $I(X)$ is $F \odot X$ in Morse's notation **(14**, p. 217**)**.) If G is such a subset of R that $E{\cdot}G \subset I(G)$, (mod $\mathbf{N}^*$), then G is called an *external $\mathfrak{D}$-open set* (with respect to $\bar{B}$ and $\mathbf{N}^*$). We use G as a generic name; $\mathbf{G}$ will denote their family. $\mathfrak{D}$ refers to Denjoy, who introduced the internal $\mathfrak{D}$-open sets under the name *ensembles-enveloppes*, for his special bases, and used them as approximation sets **(2**; 21, p. 84**)**.

From the above follows the G-pruning principle: If $\mathbf{V}$ is a $\bar{B}$-fine covering of X, and if the external $\mathfrak{D}$-open set G includes X (mod $\mathbf{N}^*$), then the family $\mathbf{V}_G$ of the $\mathbf{V}$-constituents in G is still a $\bar{B}$-fine covering of X.

Remarks. In Morse's differentiation theory **(14)**, the space R is a metric space, with, however, the slight relaxation that two distinct points may have zero distance apart. R is provided with a Carathéodory outer measure ϕ, finite on bounded sets. To each point of a subset A of R there corresponds a family $F(x)$ of sets such that every (spherical) neighborhood of x includes an $F(x)$-set. The function F is called a ***blanket.*** (Blankets are special cases of prebases **(9)**.) In all blankets studied by Morse and Hayes **(10; 11; 12; 13; 14)**, the sets occurring in the families $F(x)$ are Borelian. In order to subsume Morse's blankets under the general derivation bases, we must reduce the domain A of definition of F to the set E of points x without 0-sequences, that is, sequences $M_1, M_2, \ldots, M_i, \ldots$ with $M_i \in F(x)$, $\phi(M_i) = 0$ $(i = 1, 2, \ldots)$, which converge metrically to x; under Morse's assumptions, the set A-E of points x with 0-sequences is a ϕ-nullset **(14**, p. 218**)**. Then we correlate to each M in $F(x)$ the index $\rho = \rho(M, x) =$ diameter $(M \cup \{x\})$, and define the x-converging sequences as subsequences of the ***universal*** sequence $M_\rho\ (x)$. The restriction of ϕ to the Borelian sets is taken as our fundamental measure μ.

1.2. Comparison lemmas. For $S \subset R$, we denote by $S{\cdot}\mathbf{M}$ the family of sets $S{\cdot}M$, where $M \in \mathbf{M}$, and by μ_S the restriction of $\bar{\mu}$ to $S{\cdot}\mathbf{M}$; thus, for $M \in \mathbf{M}$,

$$\mu_S(S{\cdot}M) = \bar{\mu}(S{\cdot}M) = \mu(\bar{S}{\cdot}M).$$

For $X \subset S$, we have $\mu_S(X) = \bar{\mu}(X)$. A real-valued function h defined on S is said to be ***μ_S-measurable*** if the Lebesgue sets $[h < \alpha]$ $(-\infty < \alpha < \infty)$ belong to $S{\cdot}\mathbf{M}$.

LEMMA 1.21 We suppose that:

(A1) f and g are real-valued functions defined on P and Q, respectively, where $Q \subset P \subset R$.

(A′2) Whenever A and B are μ^*-entangled sets of finite outer measure for which $A \cup B \subset Q$, then there exist no two numbers α and β such that $\alpha < \beta$, $A \subset [f < \alpha]$, and $B \subset [g > \beta]$.

Then $f \geqslant g$ (mod $\mathbf{N}^*$) *on* Q, *that is*, $Q{\cdot}[f < g] \in \mathbf{N}^*$.

Proof. We assume the assertion to be false; thus $\bar{\mu}(Q{\cdot}[f < g]) > 0$. There exist two (rational) numbers α and β such that $\bar{\mu}(Q{\cdot}[f < \alpha < \beta < g]) > 0$. We take for A and B two equal subsets of positive finite outer measure of $[f < \alpha < \beta < g]{\cdot}Q$. Clearly $A \subset [f < \alpha]{\cdot}Q$, $B \subset [\beta < g]{\cdot}Q$, $A = B$. Since $\bar{\mu}(A) = \bar{\mu}(B) > 0$, we have a contradiction with (A′2).

LEMMA 1.22. We assume that (A1) holds and in addition:

(A″2) Whenever A and B are any two μ^*-entangled sets of finite outer measure for which $A \cup B \subset Q$, then there exist no two numbers α and β such that $\alpha < \beta$, $A \subset [f > \beta]$, and $B \subset [g < \alpha]$.

Then $f \leqslant g$ (mod $\mathbf{N}^*$) *on* Q, *that is*, $Q{\cdot}[f > g] \in \mathbf{N}^*$.

Proof. Replace f and g in Lemma 1.21 by $-f$ and $-g$, respectively.

LEMMA 1.23. We assume (A1) holds and also:

(A2) There exist no two μ^*-entangled sets A and B of finite outer measure with $A \cup B \subset Q$ such that the convex closure of $f(A)$ and $g(B)$ have positive distance apart.[3]

Then $f = g$ (mod $\mathbf{N}^*$) *on* Q, *that is*, $Q\cdot[f \neq g] \in \mathbf{N}^*$; *also the restriction* $f|Q$ *of* f *to* Q, *and* g, *are both* $\mu^*{}_Q$*-measurable.*

Proof. It is readily seen that (A′2) and (A″2) together are equivalent to (A2); application of Lemmas 1.21 and 1.22 completes the proof of the first part.

We attend to the second part. Since μ is σ-finite, $R = \bigcup R_n$, where $R_n \in \mathbf{M}$ and $\mu(R_n) < \infty$ for $n = 1, 2, \ldots$. Hence $Q = \bigcup Q_n$, where $Q_n = Q\cdot R_n$. Since the $\mu^*{}_{Q_n}$-measurability of $g|Q_n$ (restriction of g to Q_n), for all n, implies the $\mu^*{}_Q$-measurability of $g|Q = g$, we can limit ourselves to the case where $\bar{\mu}(Q)$ is finite. We assume that the $\mu^*{}_Q$-measurability of g does not hold; hence, there exists a (rational) number δ such that $D = [g < \delta]$ is not $\mu^*{}_Q$-measurable. We denote by $\bar{D}$ and $\underline{D}$ a $\mu^*{}_Q$-cover and a $\mu^*{}_Q$-kernel of D, respectively. We let $D' = D - \underline{D}$, $D'' = \bar{D} - D$. The $\mu^*{}_Q$-non-measurability of D implies that $\mu^*{}_Q(D') = \bar{\mu}(D')$ and $\mu^*{}_Q(D'') = \bar{\mu}(D'')$ are both positive. Thus, for a suitable $\beta > \delta$, the set $S = [g > \beta]\cdot D''$ is of positive outer measure. The difference

$$D^0 = \bar{D} - \underline{D} = D' + D'' \in Q \cdot \mathbf{M}^*;$$

hence there exists a $\mu^*{}_Q$-cover $\bar{S}$ of S which is included in D^0, so that $\bar{S} = \bar{S}\cdot D' + \bar{S}\cdot D''$. Since $f = g$ (mod $\mathbf{N}^*$) in Q, then $D = [f < \delta]\cdot Q$ (mod $\mathbf{N}^*$); defining $A = \bar{S}\cdot D'\cdot[f < \delta]$, then $A = \bar{S}\cdot D'$ (mod $\mathbf{N}^*$). Due to the definition of D'', $\bar{\mu}^*{}_Q(D'') = 0$, thus $\bar{S}\cdot D''$ contains no $\mu^*{}_Q$-measurable set of positive $\mu^*{}_Q$-measure. Since $\bar{S} = A + \bar{S}\cdot D''$ (mod $\mathbf{N}^*$) and $A \subset \bar{S}$, it follows that $\bar{S}$ is a $\mu^*{}_Q$-cover for A. Let $B = S$. Then A and B are $\mu^*{}_Q$-entangled, hence μ^*-entangled. If α denotes a (rational) number between δ and β, we have $A \subset [f < \alpha]$, $B \subset [g > \beta]$, contradicting (A′2), implied by (A2).

COROLLARY 1.24. *If* $P = Q = R$ (mod $\mathbf{N}^*$), (A1) *and* (A2) *imply* $f = g$ (mod $\mathbf{N}^*$) *and the* μ^**-measurability of* f *and* g.

Remarks. Lemmas 1.21 and 1.22 will be used when f is a Radon-Nikodym μ^*-integrand and g a derivate. They are analogous to de Possel's lemma (**22**, p. 394). Lemma 1.23 can be used when f and g are the (extreme) derivates. If we know somehow that both f and g are μ^*-measurable, we can formulate (A′2) and (A″2) considering only μ^*-measurable sets A and B. The μ^*-entanglement condition then means $A = B$ (mod $\mathbf{N}^*$) and $\mu^*(A) = \mu^*(B) > 0$.

[3]This formulation, which may seem unnecessarily sophisticated for numerical functions, is intended for the more general case where f and g take their values in a separable Banach space.

1.3. The individual Vitali assumption.

PRELIMINARY DEFINITIONS 1.31. By **M**-*function* we shall mean a real-valued function defined on **M**; by **M**-measure, a non-negative σ-additive **M**-function; by *signed* **M**-*measure*, a σ-additive **M**-function of variable sign.

μ-*finiteness* means finiteness on the **M**-sets of finite measure. Hence, a μ-*finite* μ-*integral* is a μ-integral $\psi(M) = \int_M f(x)\, d\mu$, finite on the **M**-sets of finite measure.

We say that the *property* (G_σ) holds if and only if R is the union of enumerably many **G**-sets G°_n such that $\bar{\mu}(G^\circ_n) < \infty$, $n = 1, 2, \ldots$.

If such a sequence G°_n exists, then a set X is said to be *bounded* if it is included in one of the sets G°_n. Thus, our notion of boundedness depends upon the special sequence of **G**-sets occurring in the formulation of (G_σ).

When (G_σ) holds, we adopt the following definitions. A *Radon* μ-*integral* is any (indefinite) μ-integral $\psi(M) = \int_M f(x)\, d\mu$, bounded in the sets G°_n; that is, there exists, for $n = 1, 2, \ldots$, a number $\beta(n)$ such that if $M \in \mathbf{M}$ and $M \subset G^\circ_n$, then $|\psi(M)| < \beta(n)$. A *Radon measure* is an **M**-measure bounded in the sets G°_n; a *signed Radon measure* is a σ-additive **M**-function bounded in the sets G°_n. A σ-*bounded function* is any real-valued function defined on R and bounded on each set G°_n.

We state some useful classical decomposition theorems. Any μ-finite signed **M**-measure is the sum of a μ-finite integral and a finite singular part. Any signed Radon measure is the sum of a Radon μ-integral and a singular part. Also, any signed Radon measure ψ is the difference of two Radon measures ψ^+ and ψ^-; the sum $\tau = \psi^+ + \psi^-$ is the *total variation* of ψ. If (G_σ) is not assumed, "Radon" can be replaced by "μ-finite".

Henceforth, when any concept involving boundedness is considered, it will be tacitly understood that (G_σ) is presupposed.

Remarks. In the formulation of Lemmas 1.21, 1.22, and 1.23, the phrase "of finite outer measure" may be replaced by "bounded," when (G_σ) holds.

In the subsequent sections we state "full differentiation theorems" for functions ψ of the type just described, namely, theorems asserting the existence almost everywhere (that is, mod $\mathbf{N}^*$) on E of the $\bar{\mathcal{B}}$-derivative $D\psi$ and its coincidence on E with a Radon-Nikodym μ^*-integrand. We avoid the use of such terms as "*R-N* derivative" (**4**, p. 133) and "pseudo-dérivée" (**22**, p. 396), reserving "derivate" and "derivative" for functions defined by means of a convergence process, either pointwise, as usual, or globally, as in (**2**) under "*L*-dérivée." In the (G_σ) case such an assertion will be proved if we establish it for any G°_n as universe and the G°_n-pruned basis as derivation basis. The sets G°_n play the part of autonomous domains of differentiation. Thus, assuming (G_σ), we reduce the case of a finite basic measure μ, in which the Radon assumption on ψ implies μ-finiteness.

We do not assume the sets G°_n to be μ- or μ^*-measurable, so the μ-covers of subsets of G°_n, in particular, of $E \cdot G^\circ_n$, need not be included in G°_n (mod $\mathbf{N}^*$)

(see Proposition 1.48). The constituents of the G°_n-pruned basis, being μ-measurable, are included (mod $\mathbf{N}^*$) in any measure kernel of G°_n.

Actually, for our purposes, a weaker form of (G_σ) suffices, as follows: There exist enumerably many sets R°_n of finite outer measure such that $R = \bigcup I(R^\circ_n)$ (mod $\mathbf{N}^*$). This property is weaker, since $I(R^\circ_n)$ need not be a $\mathbf{G}$-set. A set X is then said to be bounded if for some n, $X \subset I(R^\circ_n)$. Similarly, R°_n-pruning of a $\bar{\mathcal{B}}$-fine covering $\mathbf{V}$ of a set X means discarding all $\mathbf{V}$-sets not included in R°_n. The remaining $\mathbf{V}$-sets form a $\bar{\mathcal{B}}$-fine covering of $X \cdot I(R^\circ_n)$.

DEFINITIONS 1.32. By M-*family* we mean an enumerable family of sets, each with an associated multiplicity (**27**, p. 277). Equivalently, an M-family may be defined by any sequence of sets, the multiplicity associated with a set coinciding with its number of appearances in the sequence. In the latter formulation, abstraction is made of the order of appearance of any set. Certain advantages arise from the use of M-families instead of ordinary families in the work to follow. For instance, the *frequency* (defined a few lines farther on) is additive: thus, if $\mathbf{E}$ and $\mathbf{F}$ are M-families and $\mathbf{G}$ is the M-family obtained by uniting them, then $\phi_{\mathbf{E}} + \phi_{\mathbf{F}} = \phi_{\mathbf{G}}$. However, it is only subadditive for ordinary families. Also, any μ-measurable function on R, taking only positive integral values, may be regarded as the frequency function of a measurable M-family covering R. Awkward limitations occur if we restrict ourselves to families without repetition. In natural fashion, we may define the limit of a sequence $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n, \ldots$ of M-families as the M-family $\mathbf{E}$, if it exists, such that $\lim \phi_{\mathbf{E}_n} = \phi_{\mathbf{E}}$. So defined, $\mathbf{E}$ has an *overlap* (defined just below) which is conveniently represented by use of the Lebesgue convergence theorem.

If $\mathbf{E}$ is an M-family, then $\sigma\mathbf{E}$ will denote the union of the sets occurring in $\mathbf{E}$. By $\mathbf{E}$-*frequency* $\phi_{\mathbf{E}}(x)$ *at the point* x we shall mean the number of $\mathbf{E}$-sets (possibly ∞) to which x belongs; by $\mathbf{E}$-*excess function* we shall mean that function $\epsilon_{\mathbf{E}}$ defined on $\sigma\mathbf{E}$ by $\epsilon_{\mathbf{E}}(x) = \phi_{\mathbf{E}}(x) - 1$. We define $\theta\mathbf{E} = [\epsilon_{\mathbf{E}}(x) > 0] = [\phi_{\mathbf{E}}(x) > 1]$, and call $\theta\mathbf{E}$ the $\mathbf{E}$-*overlap set.*

Henceforth we assume that the $\mathbf{E}$-sets belong to $\mathbf{M}$. Then $\phi_{\mathbf{E}}$ and $\epsilon_{\mathbf{E}}$ are μ-measurable. If ψ is any $\mathbf{M}$-measure, we define the ψ-*overlap of* $\mathbf{E}$ by

$$\omega(\mathbf{E}, \psi) = \int_{\sigma\mathbf{E}} \epsilon_{\mathbf{E}}(x)\, d\psi.$$

In case $\psi(\sigma\mathbf{E})$ is finite, we note that

$$\omega(\mathbf{E}, \psi) = \sum\nolimits_{M \in \mathbf{E}} \psi(M) - \psi(\sigma\mathbf{E}).$$

In the particular case $\psi = \mu$, the foregoing equations define the μ-*overlap of* $\mathbf{E}$, which is of somewhat special importance (**8**, p. 193).

If $X \subset R$, M is a μ-cover for X, and ψ is any $\mathbf{M}$-measure, then the ψ-*overflow of* $\mathbf{E}$ *with respect to* X *and* M is defined as $\psi(\sigma\mathbf{E} - M \cdot \sigma\mathbf{E})$. If ψ is μ-absolutely continuous, then the quantity just defined will not depend upon the particular μ-cover M, but will be the same for every set $\bar{X}$, and the terminal phrase

"and M" may be dropped. In particular, if $\psi = \mu$, then $\mu(\sigma\mathbf{E} - \tilde{X}\cdot\sigma\mathbf{E})$ is the *μ-overflow of* $\mathbf{E}$ *with respect to* X.

If $X \subset R$, then we define the *μ-defect of covering of* X as the number $\bar{\mu}(X - X\cdot\sigma\mathbf{E})$, and we denote this by the notation $\gamma(\mathbf{E}, X, \mu)$. $\mathbf{E}$ is said to be an *ϵ-covering in measure* of X if $\gamma(\mathbf{E}, X, \mu) < \epsilon$; it is said to be an *0-covering in measure* of X if $X \subset \sigma\mathbf{E}$ (mod $\mathbf{N}^*$).

DEFINITIONS 1.33. If ψ denotes a non-negative $\mathbf{M}$-measure, we say that the basis $\tilde{\mathcal{B}}$ possesses the *Vitali ψ-property* if, and only if, for any $X \subset E$ of finite outer measure, any $\tilde{\mathcal{B}}$-fine covering $\mathbf{V}$ of X, any μ-cover M of X, and any $\epsilon > 0$, there exists an (enumerable) M-family $\mathbf{E}$ of $\mathbf{V}$-sets such that, for $S = \sigma\mathbf{E}$:

(V1) $X - X\cdot S \in \mathbf{N}^*$ ($\mathbf{E}$ is an 0-covering of X);

(V2) $\psi(S - S\cdot M) < \epsilon$ (the ψ-overflow of $\mathbf{E}$ with respect to X and M is less than ϵ);

(V3) $\omega(\mathbf{E}, \psi) < \epsilon$ (the ψ-overlap of $\mathbf{E}$ is less than ϵ).

(K. O. Househam has suggested the term *ψ-redundancy* of $\mathbf{E}$ with respect to X and M for the sum of the ψ-overflow and the ψ-overlap.)

In case only (V1) and (V3) hold, we say that $\tilde{\mathcal{B}}$ possesses the *reduced Vitali ψ-property.*

Remarks. If $\tilde{\mathcal{B}}$ possesses the Vitali property corresponding to ψ, then it evidently possesses the Vitali property corresponding to all $\psi' < \psi$; that is, the Vitali ψ-property has a hereditary character. In particular, if ψ is a Radon or a μ-finite $\mathbf{M}$-measure, $\tilde{\mathcal{B}}$ possesses the Vitali property corresponding to the μ-absolutely continuous part of ψ.

Some equivalent formulations of the Vitali ψ-property are possible. The requirement (V1) may be replaced by an ϵ-covering condition; simultaneously, "enumerable" may be replaced by "finite." That such an ϵ-covering version implies the original version can be shown by an exhaustion process. The requirement that X be of finite outer measure may be dropped. In the (G_σ) case, the phrase "of finite outer measure" may be replaced by "bounded".

DEFINITION 1.34. We define the upper *μ-approximation property of the* $\mathbf{M}$*-sets by the* $\mathbf{G}$-sets (abbreviated (UG)) as follows: Corresponding to any $\mathbf{M}$-set of M of finite measure, and any $\eta > 0$, there exists (**21**, p. 83) a $\mathbf{G}$-set G for which $M \subset G$ and $\bar{\mu}(G - M) < \eta$.

We note that (UG) implies (G_σ). (UG) is not altered if the condition "of finite measure" is waived.

PROPOSITION 1.35. *If* (UG) *holds,* ψ *is a non-negative μ-finite* (*resp., Radon*) *μ-integral, and* $\tilde{\mathcal{B}}$ *possesses the reduced Vitali ψ-property, then* $\tilde{\mathcal{B}}$ *enjoys the Vitali ψ-property.*

Proof. We let X denote any subset of E of finite outer measure (resp., bounded), $\mathbf{V}$ any $\tilde{\mathcal{B}}$-fine covering of X, ϵ any positive number. We use (UG)

to find a **G**-set $G' \supset \bar{X}$ with $\bar{\mu}(G' - \bar{X}) < 1$. Since ψ is μ-absolutely continuous and $\psi(\bar{X})$ is finite, there exists $\eta = \eta(X, \psi, \epsilon) > 0$ such that $|\psi(\bar{X}) - \psi(M)| < \epsilon$ whenever $M \in \mathbf{M}$, $M \subset G'$ and $\mu(M - \bar{X}) < \eta$, where $M - \bar{X}$ denotes Stone's difference. Invoking (UG), and the fact that the product of two $\mathfrak{D}$-open sets is again $\mathfrak{D}$-open, we find a **G**-set G with $G' \supset G \supset \bar{X}$ and $\bar{\mu}(G - \bar{X}) < \eta$. We apply the reduced Vitali ψ-property to the G-pruned family $\mathbf{V}_G$, to obtain an M-family **E** satisfying (V1) and (V3). Since the **E**-sets lie in G, we have $\mu(S - S\cdot\bar{X}) \leqslant \bar{\mu}(G - G\cdot\bar{X}) < \eta$. Thus $\psi(S) < \psi(\bar{X}) + \epsilon$, whence $\psi(S - S\cdot\bar{X}) < \epsilon$; (V2) holds, as required.

Definition 1.36. We say that *Haupt's adaptation property* holds if and only if there exists a σd-family $\mathbf{G}^\circ$ of **G**-sets which is a Borel generator for **M** (that is, **M** is the smallest $\sigma\delta$-family including $\mathbf{G}^\circ$) (**5**, p. 173).

Proposition 1.37. *Haupt's adaptation property implies the following (which includes* (UG)*): For any Radon measure ψ, any **M**-set M, and any $\epsilon > 0$, there exists a **G**-set G such that $G \supset M$ and $\bar{\psi}(G - M) < \epsilon$.*

Proof. For the case $\psi(R) < \infty$, the proof is given in (**7**, p. 27). We shall establish the theorem assuming $\psi(R) = \infty$. We introduce the sequence $G^\circ_1, G^\circ_2, \ldots, G^\circ_n, \ldots$, associated with the property (G_σ), and for any set $M \in \mathbf{M}$ and any positive integer n, we define $\psi_n(M) = \bar{\psi}(G^\circ_n \cdot M)$.

We let $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ denote a sequence of positive numbers whose sum is less than ϵ. If $M \in \mathbf{M}$, then we may apply the theorem to ψ_n, since $\psi_n(R) < \infty$, to find a $\mathbf{G}^\circ$-set G'_n such that $M \subset G'_n$ and

$$\psi_n(G'_n - M) = \bar{\psi}(G^\circ_n \cdot (G'_n - M)) < \epsilon_n.$$

The set $G^\circ_n \cdot G'_n = G''_n$ is thus a **G**-set (not necessarily a $\mathbf{G}^\circ$-set) including $G^\circ_n \cdot M$, such that $\bar{\psi}(G''_n - G^\circ_n \cdot M) < \epsilon_n$. We let S denote the union of the sets G''_n; then $S \supset M$, and

$$S - M = \bigcup G''_n - \bigcup G^\circ_n \cdot M \subset \bigcup (G''_n - G^\circ_n \cdot M),$$

hence

$$\bar{\psi}(S - M) \leqslant \sum \bar{\psi}(G''_n - G^\circ_n \cdot M) < \epsilon.$$

Since S is a **G**-set, the proposition is proved.

Remarks. The property described in Proposition 1.37 is called the *universal upper approximation property for* $\mathbf{G}^\circ$*-sets.* It holds (**16**, pp. 244–245) in the special case where R is a metric space, ψ is a classical finite Radon measure, and $\mathbf{G}^\circ$ is the family of the open sets.

Proposition 1.38. *If ψ is a μ-finite **M**-measure, Haupt's adaptation property and the reduced Vitali ψ-property both hold, then the Vitali ψ-property holds.*

Proof. This follows closely the proof of Proposition 1.35, except that we take a μ-cover M of X, and use Proposition 1.37 directly to find a **G**-set $G \supset M$ with $\bar{\psi}(G - M) < \epsilon$. As before, we find an M-family **E** satisfying

(V1) and (V3), with members lying in G. Thus $\bar{\psi}(S - S \cdot M) \leqslant \psi(G - M) < \epsilon$, and (V2) holds.

1.4. The individual full differentiation theorem for Radon or μ-finite μ-integrals.

PROPOSITION 1.41. *If ψ is a non-negative Radon (or μ-finite) μ-integral $\int f(x)\, d\mu$, and $\bar{B}$ possesses the Vitali μ-property, then $D_*\psi \geqslant f(\text{mod } \mathbf{N}^*)$ on E.*

Proof. According to the Remarks following Definition 1.31, we need treat only the case where ψ is μ-finite. We shall obtain a contradiction from the assumed existence of two μ^*-entangled subsets A and B of E of finite outer measure and two numbers α, β such that $\alpha < \beta$, $A \subset [f > \beta]$, $B \subset [g < \alpha]$, where $g = D_*\psi$. Since $[f > \beta] \in \mathbf{M}$, $A' = \bar{A} \cdot [f > \beta]$ is a μ-cover of A; since $\mu(\bar{A}) > 0$, we have

1.411 $$\psi(A') > \beta\, \mu(A').$$

On the other hand, the family $\mathbf{V}$ of the constituents V satisfying

1.412 $$\psi(V) < \alpha\, \mu(V)$$

is a $\bar{B}$-fine covering of $B \subset [g < \alpha]$. Thus, by virtue of the Vitali μ-property, for any natural number n, there exists an M-family $\mathbf{E}_n$ of $\mathbf{V}$-sets V_{ni}, such that if $S_n = \sigma \mathbf{E}_n$, then

1.413 $B - B \cdot S_n \in \mathbf{N}^*$; $\mu(S_n - S_n \cdot \bar{B}) < 2^{-n}$; μ —overlap of $\mathbf{E}_n$ is less than 2^{-n}.

Using 1.413 and 1.412 we obtain

1.414 $$\psi(\bar{B}) \leqslant \psi(S_n) \leqslant \sum_i \psi(V_{ni}) < \alpha \sum_i \mu(V_{ni}) < \alpha(\mu(S_n) + 2^{-n}),$$

and $\lim \mu(S_n) = \mu(\bar{B})$. Combining, we obtain

1.415 $$\psi(\bar{B}) \leqslant \alpha\mu(\bar{B}),$$

which, since $\psi(A') = \psi(\bar{B})$ and $\alpha < \beta$, is a contradiction of 1.411. From Lemma 1.22 follows the assertion $f \leqslant g$ (mod $\mathbf{N}^*$).

Remarks. If an ϵ-covering version of the Vitali μ-property is used in place of the 0-covering version, then the first statement in 1.413 is replaced by $\mu(\bar{B} - \bar{B} \cdot S_n) < \eta_n$, and because of the μ-absolute continuity of ψ, η_n can be so chosen that $\psi(\bar{B}) \leqslant \psi(S_n) + 2^{-n}$; 1.414 has to be altered accordingly.

An example of a blanket possessing the Vitali μ-property, and a function $g \in \mathfrak{L}^{(p)}$ for every $p > 1$, such that its integral ψ has $D\psi = \infty$ everywhere, is known (**11**, p. 293).

PROPOSITION 1.42. *If ψ is a non-negative Radon (or μ-finite) μ-integral $\int f(x)\, d\mu$, and $\bar{B}$ possesses the Vitali ψ-property, then $D^*\psi \leqslant f(\text{mod } \mathbf{N}^*)$ on E.*

Proof. As in the preceding proposition, we may and do assume that ψ is μ-finite. We assume that A and B are two μ^*-entangled sets of finite outer measure, α and β two numbers such that $\alpha < \beta$, $A \subset [f < \alpha]$, $B \subset [g > \beta]$,

where $g = D^*\psi$. Since $[f < \alpha] \in \mathbf{M}$, $A' = \bar{A}\cdot[f < \alpha]$ is a μ-cover for A. Since $\mu(\bar{A}) > 0$, we obtain

1.421 $$\psi(A') < \alpha\mu(A').$$

The family $\mathbf{V}$ of the constituents V satisfying

1.422 $$\psi(V) > \beta\mu(V)$$

is a $\bar{B}$-fine covering of $B \subset [g > \beta]$. We use the Vitali ψ-property to determine for each natural number n an M-family $\mathbf{E}_n$ of $\mathbf{V}$-sets V_{ni} such that if $S_n = \sigma\mathbf{E}_n$, then

1.423 $B - B\cdot S_n \in \mathbf{N}^*$; $\psi(S_n - S_n\cdot\bar{B}) < 2^{-n}$; ψ —overlap of $\mathbf{E}_n$ is less than 2^{-n}.

The ψ-overlap condition yields

$$\psi(S_n) > \sum_i \psi(V_{ni}) - 2^{-n};$$

hence, using 1.422,

$$\psi(S_n) > \beta\sum_i \mu(V_{ni}) - 2^{-n} \geqslant \beta\,\mu(S_n) - 2^{-n}.$$

This last and 1.423 together yield, for $n = 1, 2, \ldots,$

1.424 $$\psi(\bar{B}) + 2^{-n} > \beta\mu(\bar{B}) - 2^{-n};$$

hence $\psi(\bar{B}) \geqslant \beta\mu(\bar{B})$, which contradicts 1.421, since $\alpha < \beta$ and $\psi(A') = \psi(\bar{B})$. Thus, Lemma 1.21 applies and $D^*\psi \leqslant f$ (mod $\mathbf{N}^*$) on E.

Remark. With the ϵ-covering version of the Vitali ψ-property, we replace the first statement in 1.423 by $\mu(\bar{B} - \bar{B}\cdot S_n) < 2^{-n}$, hence $\mu(S_n) > \mu(\bar{B}) - 2^{-n}$, and in 1.424, we replace $\mu(\bar{B})$ by $\mu(\bar{B}) - 2^{-n}$.

THEOREM 1.43. *If ψ is a non-negative Radon (or μ-finite) μ-integral and $\bar{B}$ possesses the Vitali μ-property and the Vitali ψ-property, then the $\bar{B}$-derivative $D\psi$ exists almost everywhere on E and is equal,* mod $\mathbf{N}^*$, *to $f|E$, where f denotes any Radon-Nikodym μ^*-integrand of ψ.*

Proof. This is an immediate consequence of Propositions 1.41 and 1.42.

DEFINITION 1.44. An $\mathbf{M}$-function ψ is said to be *majorized* or *dominated* by the $\mathbf{M}$-function ψ° if $|\psi(M)| \leqslant \psi^\circ(M)$ for every $M \in \mathbf{M}$.

We note that a signed Radon measure (resp., μ-finite $\mathbf{M}$-measure) dominated by a μ-integral is itself a μ-integral; also a finitely additive $\mathbf{M}$-function dominated by a Radon measure (resp., μ-finite $\mathbf{M}$-measure) is a signed Radon measure (resp., signed μ-finite $\mathbf{M}$-measure).

THEOREM 1.45. *If ψ° is a non-negative Radon (or μ-finite) μ-integral and $\bar{B}$ possesses the Vitali μ-property and the Vitali ψ°-property, then for any signed Radon measure (or μ-finite signed* $\mathbf{M}$*-measure) ψ dominated by ψ°; the $\bar{B}$-derivative $D\psi$ exists* p.p. *on E and is equal to the E-restriction of a Radon-Nikodym integrand of ψ.*

Proof. This follows immediately upon decomposing ψ into ψ^+ and ψ^- and using the hereditary character of the Vitali ψ-property.

THEOREM 1.46. *If ψ° is a non-negative Radon μ-integral, $\bar{B}$ possesses the Vitali μ-property and the Vitali ψ°-property, ψ is a signed Radon measure, and there corresponds to each G°_n a positive finite number $\kappa(n)$ such that $|\psi(M)| < \kappa(n)\cdot\psi^\circ(M)$ for any* **M**-*set $M \subset G^\circ_n$ (ψ°-Lipschitz condition), then $D\psi$ exists almost everywhere on E and is equal to the E-restriction of a Radon-Nikodym integrand of ψ.*

Proof. Apply Theorem 1.45 to each G°_n used as an autonomous domain of $\bar{B}$-differentiation, with $\kappa(n)\cdot\psi^\circ$ as majorant.

Remark. If we know that the extreme derivates are μ^*-measurable, then from the Remarks under Corollary 1.24, it follows that Theorems 1.43, 1.45, and 1.46 remain valid, if, in the definition of the Vitali property, X is taken from **M**.

DEFINITION 1.47. The special case of the Vitali property, wherein $\psi = \mu$, is the so-called ***weak Vitali property***, and a basis $\bar{B}$ possessing it is called a ***weak*** derivation basis.

Remarks. By Theorem 1.46, such a basis differentiates (in de Possel's sense) the uniformly μ-Lipschitzian integrals; explicitly, if ψ is a σ-additive **M**-function for which $|\psi(M)| < \kappa\, \mu(M)$ for $M \in$ **M**, where κ is a constant, then $D\psi$ exists almost everywhere on E and is equal to the E-restriction of a Radon-Nikodym integrand of ψ. In de Possel's version, a weak derivation basis differentiates the integral of any essentially bounded μ-measurable function, and in the Radon case, the integrals of functions which are measurable and essentially bounded on each G°_n. Five equivalent properties defining these bases in the case $E = R$ (mod **N***) are given in (**22**, pp. 403–405).

PROPOSITION 1.48. *If each B-fine covering of any subset X of E admits an enumerable subfamily covering X* (mod **N***), *then for any* **G**-*set G, we have $\overline{E\cdot G} \subset G$* (mod **N***) *or equivalently $E\cdot G \subset \underline{G}$* (mod **N***). *If in addition, $E = R$, then the* **G**-*sets are μ^*-measurable.*

Proof. The family **W** of the B-constituents included in G is a B-fine covering of $E\cdot G$; thus there exists an M-family $\mathbf{E} \subset \mathbf{W}$ with $E\cdot G \subset \sigma\mathbf{E}$ (mod **N***). Hence $\overline{E\cdot G} \subset \sigma\mathbf{E}$ (mod **N**), and $\overline{E\cdot G} \subset G$ (mod **N***). If $E = R$, then $\bar{G} \subset G$ (mod **N***), hence $G = \bar{G}$ (mod **N***).

1.5. The individual full differentiation theorem for Radon measures.

LEMMA 1.51. *If ψ is a Radon measure (or a μ-finite* **M**-*measure), $\bar{B}$ possesses the Vitali ψ-property, $Q \subset E$, $\mu(Q) < \infty$, $0 < \eta < \infty$, and there exists a $\bar{B}$-fine covering* **V** *of Q such that for all* **V**-*sets V,*

1.511 $$\psi(V) > \eta\, \mu(V)$$

then $\psi(M) > \eta\, \bar{\mu}(Q)$ for any $M \in$ **M** *with $Q \subset M$.*

Proof. The Remarks following Definitions 1.31 permit us to consider only the case of a μ-finite **M**-measure. We may also assume $\mu(M) < \infty$. We let ϵ denote an arbitrary positive number, let T denote a μ-cover of Q for which $Q \subset T \subset M$, and invoke the Vitali ψ-property to obtain an M-family **E** of sets V_i, $i = 1, 2, \ldots$, for which, putting $\sigma\mathbf{E} = S$, we have

1.512 $\quad Q - Q{\cdot}S \in \mathbf{N}^*$; ψ-overlap of **E** is less than ϵ; $\psi(S - S{\cdot}T) < \epsilon$.

From 1.511 and the first two conditions of 1.512 we obtain

$$\psi(S) > \sum_i \psi(V_i) - \epsilon \geqslant \eta \cdot \sum_i \mu(V_i) - \epsilon \geqslant \eta\, \mu(S) - \epsilon \geqslant \eta\bar{\mu}(Q) - \epsilon;$$

this result, combined with the last inequality of 1.512, yields

$$\psi(M) \geqslant \psi(T) \geqslant \psi(T{\cdot}S) > \psi(S) - \epsilon \geqslant \eta\, \bar{\mu}(Q) - 2\epsilon,$$

which, since ϵ is arbitrary, gives the desired relation.

THEOREM 1.52. *If ψ is a Radon measure (or a μ-finite* **M**-*measure) and $\bar{B}$ possess the Vitali μ- and ψ-properties, then ψ possesses almost everywhere in E a $\bar{B}$-derivative $D\psi$ which is equal to a Radon-Nikodym integrand of ψ.*

Proof. We decompose ψ into the μ-regular part ψ_r and the μ-singular part ψ_s, denoting by N_0 an **N**-set on which ψ_s is concentrated, that is, $\psi_s(R-N_0)=0$. $\bar{B}$ possesses the Vitali μ- and ψ-properties, hence, in accordance with the Remarks under Definitions 1.33, $\bar{B}$ also has the Vitali μ- and ψ_r-properties. Due to Theorem 1.43, we need prove only that $D^*\psi_s = 0$ (mod $\mathbf{N}^*$).

We let $A_n = [D^*\psi_s > n^{-1}]{\cdot}(R - N_0)$. The family of $\bar{B}$-constituents V for which $\psi_s(V) \geqslant n^{-1}\,\mu(V)$ is a $\bar{B}$-fine covering of A_n. In accordance with Lemma 1.51, $\psi_s(M) \geqslant n^{-1}\,\bar{\mu}(A_n)$ for any $M \in \mathbf{M}$ with $A_n \subset M$; in particular, this holds for $M = R - N_0$, thus $0 \geqslant n^{-1}\,\bar{\mu}(A_n)$, hence $\bar{\mu}(A_n) = 0$, and A_n is an $\mathbf{N}^*$-set. Now

$$[D^*\psi_s > 0]\cdot(R - N_0) = \bigcup_n [D^*\psi_s > n^{-1}]\cdot(R - N_0) = \bigcup_n A_n,$$

which is therefore also an $\mathbf{N}^*$-set. Finally,

$$[D^*\psi_s > 0] = [D^*\psi_s > 0]{\cdot}N_0 + [D^*\psi_s > 0]{\cdot}(R - N_0)$$

is an $\mathbf{N}^*$-set, and the proof is complete.

The following results are immediate consequences of Theorems 1.52, 1.45, and 1.46.

THEOREM 1.53. *If ψ° is a Radon measure (resp., μ-finite* **M**-*measure) and $\bar{B}$ possesses the Vitali μ- and ψ°-properties, then $\bar{B}$ differentiates any signed Radon (resp., μ-finite)* **M**-*measure dominated by ψ°.*

THEOREM 1.54. *If ψ° is a Radon measure, $\bar{B}$ possesses the Vitali μ- and ψ°-properties, and ψ is such a signed Radon measure that there corresponds to any*

G°_n a positive (finite) number $\kappa(n)$ such that $|\psi(M)| < \kappa(n)\cdot\psi^\circ(M)$ for any **M***-set $M \subset G^\circ_n$, then $\bar{B}$ differentiates ψ.*

The remark following Theorem 1.46 applies here also.

1.6. Class differentiation theorems.

DEFINITION 1.61. If $\bar{B}$ possesses the Vitali ψ-property for *every* non-negative Radon (resp., μ-finite) μ-integral ψ, then we say that $\bar{B}$ has the *Vitali property for non-negative Radon* (resp., *μ-finite*) *μ-integrals.*

THEOREM 1.62. *If $\bar{B}$ has the Vitali property for non-negative Radon* (*resp., μ-finite*) *μ-integrals, then $\bar{B}$ differentiates every Radon* (*resp., μ-finite*) *μ-integral.*

Proof. This follows from Theorem 1.43.

DEFINITION 1.63. If $\bar{B}$ possesses the Vitali ψ-property for *every* Radon (resp., μ-finite) **M**-measure ψ, then we say that $\bar{B}$ has the *Vitali property for Radon* (resp., *μ-finite*) **M**-measures.

THEOREM 1.64. *If $\bar{B}$ has the Vitali property for Radon* (*resp., μ-finite*) **M***-measures, then $\bar{B}$ differentiates every Radon* (*resp., μ-finite*) **M***-measure.*[4]

Proof. This follows from Theorem 1.52.

Remark. De Possel (**23**; **24**) defines a "système dérivant généralisé" as a correspondence to each point x of a filter F_x of non-negative μ-measurable summable real functions f, vanishing outside a measurable set of finite measure (depending on f), and with $\int_R f d\mu > 0$, ψ denotes any function on **M** into a Banach space, enumerably additive and of bounded variation. The derivative $D\psi(x)$ is defined as

$$\lim_{F_x}\left(\int_R f d\psi \Big/ \int_R f d\mu\right).$$

Conditions are stated for F_x to differentiate Lipschitz, μ-absolutely continuous, and general functions ψ.

DEFINITIONS 1.65. We shall introduce a chain of properties between the Vitali μ-property and the Vitali property for non-negative Radon μ-integrals, under the assumption that (G_σ) holds. We let p and q denote two numbers, both greater than 1, for which $p^{-1} + q^{-1} = 1$. By *$\mu^{(q)}$-functions* we shall mean those Radon μ-integrals ψ of the form $\psi(M) = \int_M f(x)\, d\mu$, for $M \in$ **M**, where f is such a function that for any given positive integer n, $\int_M |f(x)|^q d\mu$ is (uniformly) bounded on the **M**-sets M included in G°_n; by $\mathfrak{L}^{(q)}$-functions we shall mean those functions f which are integrands of $\mu^{(q)}$-functions. We shall say that $\bar{B}$ is an *$S^{(p)}$-basis* if and only if for each subset $X \subset E$ of finite outer

[4]In case $\bar{B}$ is a blanket, the Vitali property for classical Radon measures is the "pseudo-strength" of (**10**), which also is referred to as "Vitalische-Hayes'sche Eigenschaft" in (**21**, p. 91). The existence p.p. of the derivative of any classical Radon measure is established in (**10**).

measure, each $\bar{B}$-fine covering $\mathbf{V}$ of X, and each $\epsilon > 0$, there exists an M-family $\mathbf{E}$ of V-sets for which, putting $\sigma\mathbf{E} = S$,

(I) $\mathbf{E}$ is an 0-covering of X;

(II) the μ-overflow of $\mathbf{E}$ with respect to X is less than ϵ;

(III) $\int_S\{\epsilon_{\mathbf{E}}(x)\}^p d\mu < \epsilon$ (the $\mathfrak{L}^{(p)}$-*overlap* of $\mathbf{E}$ is less than ϵ).

Statements (I), (II), and (III) are meaningful for $p = 1$; we accordingly define an $S^{(1)}$-*basis* as one having properties (I), (II), and (III), with $p = 1$. We define as μ^∞-*functions* all integrals of μ-measurable functions which are essentially bounded on each set G°_n.

Remarks. Comparison with Definitions 1.33 shows that for any $p > 1$, (I) and (II) are the same as (V1) and (V2), while (III) is at least as strong as (V3) for $\psi = \mu$; hence every $S^{(p)}$-basis, $p > 1$, possesses the Vitali μ-property, and, in accordance with the Remarks following Definition 1.47, differentiates the μ^∞-functions. The following is an extension of this result.

THEOREM 1.66. *If $p > 1$ and $\bar{B}$ is an $S^{(p)}$-basis, then $\bar{B}$ differentiates the $\mu^{(q)}$-functions.*

Proof. We let $\bar{B}$ denote any $S^{(p)}$-basis. From the property (G_σ), it follows that we may restrict our proof to the case where the domain E of $\bar{B}$ lies within one set G°_N; that is, E may be assumed to be bounded. Furthermore, it follows from the remarks just above, and from Theorem 1.45, that we need prove only that for each non-negative $\mu^{(q)}$-function ψ, defined by $\psi(M) = \int_M f(x)\cdot d\mu$ for $M \in \mathbf{M}$, $\bar{B}$ possesses the Vitali ψ-property.

Accordingly, we let X denote any subset of E (necessarily of finite outer measure), $\mathbf{V}$ any $\bar{B}$-fine covering of X, and ϵ any positive number. We put $G^\circ_N = G$ and define ϵ' as any positive number such that

1.661 $$(\epsilon')^{1/p}\left(\int_{\underline{G}}\{f(x)\}^q\cdot d\mu\right)^{1/q} < \epsilon.$$

From the μ-absolute continuity of ψ on the M-subsets of G, it follows that there exists a positive number η for which

1.662 $$|\psi(M') - \psi(M'')| < \epsilon$$

whenever $\mu(M' - M'') < \eta$, where $M' \in \mathbf{M}$, $M'' \in \mathbf{M}$, $M' \subset G$, $M'' \subset G$, and $M' - M''$ denotes Stone's difference. We may and do assume that $\eta < \epsilon'$.

We may assume $\mathbf{V}$ to be G-pruned. We use the $S^{(p)}$-properties of $\bar{B}$ to find an M-family $\mathbf{E}$ of V-sets for which, putting $S = \sigma\mathbf{E} \subset G$, we have

1.663 $$X - X\cdot S \in \mathbf{N}^*;\ \mu(S - S\cdot\bar{X}) < \eta;\ \int_S\{\epsilon_{\mathbf{E}}(x)\}^p\cdot d\mu < \eta.$$

Evidently $\mathbf{E}$ satisfies (V1) of Definition 1.33. From the first relation in 1.663 and Proposition 1.48, we see that $S \subset \underline{G}$ (mod $\mathbf{N}^*$) and $\bar{X} \subset G$ (mod $\mathbf{N}^*$); hence, because of 1.662,

$$|\psi(S) - \psi(\bar{X}\cdot S)| < \epsilon;$$

therefore $\psi(S - S\cdot\bar{X}) < \epsilon$, and (V2) holds.

Using Hölder's inequality, 1.661, and the last relation in 1.663, we have

$$\begin{aligned}\int_{\sigma E} \epsilon_E(x)\, d\psi &= \int_S \epsilon_E(x) f(x)\, d\mu \\ &< \left(\int_S \{\epsilon_E(x)\}^p d\mu\right)^{1/p} \left(\int_S \{f(x)\}^q d\mu\right)^{1/q} \\ &< \eta^{1/p} \left(\int_{\underline{G}} \{f(x)\}^q d\mu\right)^{1/q} < \epsilon.\end{aligned}$$

Hence (V3) holds, and the proof is complete.

Remark. In **(11)**, there is given an example of an $S^{(p)}$-basis ($p > 1$) and a function which is a $\mu^{(q')}$-function for each q', $q' < p/(p-1)$, whose derivative is infinite everywhere. In this example, as in all counter-examples known to us in the theory of differentiation, a derivate is infinite on a set of positive measure. In this connection it is interesting to observe that Zygmund's proof **(29)** depends upon the summability of the derivates, which prevents a "flight to infinity" on a set of positive measure.

THEOREM 1.67. *In the definition of an $S^{(p)}$-basis, the* 0*-covering condition may be replaced by an ϵ-covering condition; simultaneously* **E** *may be required to be finite.*

Proof. Since we are merely relaxing the initial definition of an $S^{(p)}$-basis, we have to prove only that any $S^{(p)}$-basis under the ϵ-covering definition is an $S^{(p)}$-basis under the 0-covering definition. We thus assume that for any subset $X \subset E$ of finite outer measure, any $\bar{B}$-fine covering **V** of X, and any $\epsilon > 0$, there exists a finite family **F** of **V**-constituents such that

1.671 $$\bar{\mu}(X - X \cdot \sigma\mathbf{F}) < \epsilon;\ \mu(\sigma\mathbf{F} - \bar{X} \cdot \sigma\mathbf{F}) < \epsilon;\ \int_{\sigma\mathbf{F}} \{\epsilon_{\mathbf{F}}(x)\}^p d\mu < \epsilon.$$

We take a subset X of E of finite outer measure, a $\bar{B}$-fine covering of X, and a positive number ϵ. We choose a sequence of positive numbers, $\eta_1, \eta_2, \ldots, \eta_n, \ldots$ whose sum is less than ϵ.

We shall determine inductively a sequence of finite families $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n, \ldots$ of **V**-constituents, such that, for $n = 1, 2, \ldots$:

(a) $\mathbf{F}_1 \subset \mathbf{F}_2 \subset \ldots \subset \mathbf{F}_n$;

(b) $\mu(\bar{X} - \bar{X} \cdot \sigma\mathbf{F}_n) < \eta_n$;

(c) $\mu(\sigma\mathbf{F}_n - \bar{X} \cdot \sigma\mathbf{F}_n) < \sum_{i=1}^{n} \eta_i = \zeta_n$;

(d) $\int_{\sigma\mathbf{F}_n} \{\epsilon_{\mathbf{F}_n}(x)\}^p d\mu < \sum_{i=1}^{n} \eta_i = \zeta_n$.

The existence of a family $\mathbf{F}_1$ satisfying (b), (c), and (d), for $n = 1$, follows from our hypotheses as expressed in 1.671. We assume the existence of a nested sequence of families $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n$, satisfying (a), (b), (c), and (d), and proceed to find $\mathbf{F}_{n+1}$ also satisfying them.

We put $\sigma\mathbf{F}_n = S$, $X - X\cdot S = Y$; then $\bar{Y} = \bar{X} - \bar{X}\cdot S$ is a μ-cover for Y. From (d) and the fact that $\mu(S) < \infty$ it follows that

$$\int_S \{\phi_{\mathbf{F}_n}(x)\}^p d\mu < \infty;$$

thus we may find a positive number $\gamma = \gamma(\eta_{n+1})$ such that

1.672 $$\int_M \{\phi_{\mathbf{F}_n}(x)\}^p d\mu < \eta_{n+1}/2^p$$

whenever M is an **M**-set, $M \subset S$, and $\mu(M) < \gamma$. We may and do assume that $\gamma < \eta_{n+1}/2^p$.

Again recalling 1.671, we find a finite subfamily **H** of **V** for which, putting $\sigma\mathbf{H} = T$,

1.673 $$\mu(\bar{Y} - \bar{Y}\cdot T) < \gamma;\ \mu(T - \bar{Y}\cdot T) < \gamma;\ \int_T \{\epsilon_{\mathbf{H}}(x)\}^p d\mu < \gamma.$$

Noting that $S\cdot T \subset T - \bar{Y}\cdot T$, using 1.672, and the second relation of 1.673, we obtain

1.674 $$\int_{S\cdot T} \{\phi_{\mathbf{F}_n}(x)\}^p d\mu < \eta_{n+1}/2^p.$$

We define $\mathbf{F}_{n+1} = \mathbf{F}_n \cup \mathbf{H}$ and let $\sigma\mathbf{F}_{n+1} = U$. We observe that $\bar{X} - \bar{X}\cdot U = \bar{Y} - \bar{Y}\cdot T$ and $(U - \bar{X}\cdot U) \subset (S - \bar{X}\cdot S) + (T - \bar{Y}\cdot T)$, whence from 1.673 and (c), we obtain

$$\mu(\bar{X} - \bar{X}\cdot U) < \gamma < \eta_{n+1};\ \mu(U - \bar{X}\cdot U) < \zeta_n + \gamma < \zeta_n + \eta_{n+1} = \sum_{i=1}^{n+1} \eta_i,$$

which establishes (b) and (c) as applied to $\mathbf{F}_{n+1}$.

Next,

1.675 $$\int_U \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu = \int_{U-T} \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu + \int_T \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu.$$

Since $U - T \subset S$, and $\epsilon_{\mathbf{F}_{n+1}} = \epsilon_{\mathbf{F}_n}$ on $U - T$, then by (d),

1.676 $$\int_{U-T} \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu < \int_S \{\epsilon_{\mathbf{F}_n}(x)\}^p d\mu < \zeta_n.$$

Using 1.673, 1.674, and Minkowski's inequality, we also obtain

$$\begin{aligned}\int_T \{\epsilon_{\mathbf{F}_{n+1}}(x)\}^p d\mu &= \int_T \{\phi_{\mathbf{F}_n}(x) + \phi_{\mathbf{H}}(x) - 1\}^p d\mu \\ &< \left[\left(\int_T \{\phi_{\mathbf{F}_n}(x)\}^p d\mu\right)^{1/p} + \left(\int_T \{\phi_{\mathbf{H}}(x) - 1\}^p d\mu\right)^{1/p}\right]^p \\ &= \left[\left(\int_{T\cdot S} \{\phi_{\mathbf{F}_n}(x)\}^p d\mu\right)^{1/p} + \left(\int_T \{\epsilon_{\mathbf{H}}(x)\}^p d\mu\right)^{1/p}\right]^p < \eta_{n+1}.\end{aligned}$$

Putting this last inequality and 1.676 into 1.675, we establish (d) as applied to $\mathbf{F}_{n+1}$.

Finally, we define

$$\mathbf{E} = \bigcup_{n=1}^{\infty} \mathbf{F}_n.$$

It is clear from (a), (b), (c), and (d) that $\mathbf{E}$ is an M-family of $\mathbf{V}$-constituents satisfying conditions (I), (II), and (III) of Definitions 1.65, which completes our proof.

Remarks. No change in the definition of an $S^{(p)}$-basis results if X is required to be a bounded subset of E. For then any subset $Y \subset E$ of finite outer measure may be decomposed into a countable sum of bounded disjoint subsets of E, for each of which the $S^{(p)}$-property holds, and by the method of the preceding theorem a countable family $\mathbf{F}$ of $\mathbf{V}$-sets may be found which is an 0-covering of Y, with $\mathfrak{L}^{(p)}$-overlap and (μ, Y)-overflow of $\mathbf{F}$ both as small as desired.

If, in the proof of Theorem 1.66, we adopt the ϵ-covering version for the definition of $S^{(p)}$-bases, the first relation in 1.663 is replaced by $\mu(\tilde{X} - \tilde{X}\cdot S) < \epsilon$. The Vitali ψ-property is established in the ϵ-covering version.

1.7. Relation to Younovitch's differentiation theorem. Younovitch (**28**, Theorem III) assumes that $E = R$, μ is complete, and $\mu(R)$ is finite. The sets consisting of a single point belong to $\mathbf{M}$. The basis $\tilde{B}$ is a special de Possel basis such that each point x there corresponds to one ordinary sequence $A_1(x), A_2(x), \ldots, A_n(x), \ldots$ of "neighborhoods" such that $x \in A_n(x)$. The x-converging sequences are the subsequences of the basic sequence $A_1(x)$, $A_2(x), \ldots, A_n(x), \ldots$.

YOUNOVITCH'S DIFFERENTIATION THEOREM 1.71. *$\tilde{B}$ differentiates the (finite) μ-integrals if there exists a positive constant α such that:*

(Y1) *Corresponding to any set M in* $\mathbf{M}$ *of positive μ-measure, any $\tilde{B}$-fine covering* $\mathbf{V}$ *of M, and any posiive ϵ, there exists a finite disjoint subfamily V_1, $V_2, \ldots, V_n$, satisfying the inequalities*

1.711 $$\sum_{i=1}^{n} \mu(V_i - V_i \cdot M) < \epsilon, \quad \mu\left(M \cdot \sum_{i=1}^{n} V_i\right) > \alpha\mu(M).$$

In addition, each set V_i, $i = 1, 2, \ldots, n$, must belong to the basic sequence corresponding to a point p_i of M; that is, for a certain index n_i,

$$V_i = A_{n_i}(p_i).$$

The second inequality of 1.711 may be expressed by saying that the *exhaustion power* of the V_i with respect to M is greater than α.

Remarks. If ψ denotes a non-negative μ-integral, (Y1) holds, and M, $\mathbf{V}$, and ϵ are regarded as prescribed, then we can, by suitable finite iteration of the exhaustion process under ψ-overflow control, produce a finite system $q_1, \ldots, q_k$ of points of M and indices $n_1, \ldots, n_k$ such that

(1) the μ-defect of covering of M by the sets

$$A^j = A_{n_j}(q_j)$$

is less than ϵ;

(2) the ψ-excess of covering of M by the sets A^j is less than ϵ;

(3) the ψ-overlap of the sets A^j is less than ϵ.

Consequently, (Y1) implies the Vitali property for integrals, (i) expressed in the ϵ-covering version, (ii) with reference to anchorage points, (iii) restricted to **M**-measurable sets. As noted previously, the ϵ-covering version with finite M-family **E** is equivalent to our original one. Actually, by infinite iteration of the exhaustion process outlined above, we can obtain an (enumerable) M-family satisfying (2) and (3), and covering M (mod **N**).

Younovitch, like de Possel, formulates Vitali-covering properties with reference to points, thus strengthening the Vitali assumptions. Condition (iii), however, appears to be substantially weaker than the Vitali property for μ-integrals. If, by any means, one can prove under Younovitch's assumptions that the (extreme) derivates are μ-measurable (see Remark after Theorem 1.46), or that any $\bar{\mathfrak{B}}$-fine covering of a set X is a $\bar{\mathfrak{B}}$-fine covering of any measure cover for X, then Younovitch's theorem follows from Theorem 1.62. Younovitch's note, unfortunately, contains neither proofs nor even a hint of them.

§2. The Converse Problem; Covering Properties deduced from Differentiation Properties of σ-Additive Set Functions

2.1. De Possel's equivalence theorem. (22, pp. 403–405).

Definition 2.11. A derivation basis $\bar{\mathfrak{B}}$ possesses the *density property* if it differentiates the integrals of the characteristic functions of μ-measurable sets, that is, if, for any **M**-set M, the density of x, defined as the limit of $\mu(M_\iota(x)\cdot M)/\mu(M_\iota(x))$, exists and equals C_M (characteristic function of M) almost everywhere on E.

Theorem 2.12. *The density property and the Vitali μ-property are equivalent.*

Proof. As noted in the Remarks after Definition 1.47, the Vitali μ-property implies the density property. We have to prove the converse.

We assume that the density property holds and let X denote a subset of E of finite outer measure, **V** a $\bar{\mathfrak{B}}$-fine covering of X, and ϵ a positive number. We select α, $0 < \alpha < 1$, so that

2.121 $$0 < (\alpha^{-1} - 1)\,\mu(\tilde{X}) < \epsilon.$$

If Y is any subset of X such that $\bar{\mu}(Y) > 0$, then we define $\mathbf{V}(Y, \alpha)$ as the family of **V**-sets V for which

2.122 $$\mu(\tilde{Y}\cdot V) > \alpha\mu(V),$$

and μ_Y as the supremum of the numbers $\mu(V)$, for $V \in \mathbf{V}(Y, \alpha)$.

Now, from the density property, if $\bar{\mu}(Y) > 0$, it follows that the density of Y equals 1 for at least one point $y \in Y$. Hence there exists at least one y-converging sequence in $\bar{B}$ whose constituents belong to $\mathbf{V}$ and satisfy 2.122. The family $\mathbf{V}(Y, \alpha)$ is thus non-vacuous, hence $\mu_Y > 0$. In case $Y \subset X$ and $\bar{\mu}(Y) = 0$, we put $\mu_Y = 0$.

We fix a number κ, $0 < \kappa < 1$. By the definition of μ_X, there exists a $\mathbf{V}$-set V_1, such that

$$\mu(V_1) > \kappa\,\mu_X, \quad \mu(\hat{X}\cdot V_1) > \alpha\mu(V_1).$$

We let $X_1 = X$, $X_2 = X_1 - V_1 \cdot X_1$. From this point we proceed inductively, assuming that sets V_i have been defined in $\mathbf{V}$ for $i = 1, 2, \ldots, n$, satisfying the relations

2.123 $$\mu(\hat{X}_i \cdot V_i) > \alpha\mu(V_i), \ \mu(V_i) > \kappa\mu_{X_i},$$

where

$$X_{i+1} = X_1 - X_1 \cdot \bigcup_{j=1}^{i} V_j.$$

In case $\bar{\mu}(X_{n+1}) = 0$, we stop the process; in case $\bar{\mu}(X_{n+1}) > 0$, we define a new $\mathbf{V}$-constituent V_{n+1} such that

$$\mu(V_{n+1}) > \kappa\,\mu_{X_{n+1}}, \quad \bar{\mu}(X_{n+1} \cdot V_{n+1}) > \alpha\mu(V_n).$$

The process just described leads to the construction of an M-family $\mathbf{E}$ consisting of a finite or infinite sequence of sets V_i taken from $\mathbf{V}$, satisfying 2.123 ($i = 1, 2, \ldots$). Since also the sets $X_i \cdot V_i$ are disjoint (mod $\mathbf{N}^*$), we have

$$\begin{aligned} \mu(\hat{X}) > \mu\Big(\hat{X} \cdot \bigcup_i V_i\Big) > \mu\Big(\bigcup_i \hat{X}_i \cdot V_i\Big) \\ = \sum_i \mu(\hat{X}_i \cdot V_i) > \alpha \sum_i \mu(V_i); \end{aligned}$$

consequently,

2.124 $$\sum_i \mu(V_i) < \alpha^{-1}\mu\Big(\hat{X} \cdot \bigcup_i V_i\Big).$$

Putting $S = \sigma\mathbf{E}$ and combining 2.121 with 2.124, we obtain

2.125 $$\begin{aligned} \Big\{\sum_i \mu(V_i) - \mu(S)\Big\} + \mu(S - S \cdot \hat{X}) = \sum_i \mu(V_i) - \mu(S \cdot \hat{X}) \\ < (\alpha^{-1} - 1)\,\mu(S \cdot \hat{X}) < \epsilon. \end{aligned}$$

Hence conditions (V2) and (V3) of Definition 1.33, with $\psi = \mu$, hold.

To show that (V1) holds for our family $\mathbf{E}$, we note that if the sequence of sets V_i is finite, then for some positive integer N, we have $\mathbf{E} = \{V_1, V_2, \ldots, V_N\}$ and $\mu(\hat{X}_{N+1}) = 0$. Thus $\bar{\mu}(X - X \cdot \sigma\mathbf{E}) = 0$, as required by (V1). If the sequence is infinite, then from 2.123 and 2.124 we see that

$$\kappa \sum_i \mu_{X_i} < \sum_i \mu(V_i) < \alpha\mu(\hat{X}) < \infty; \text{ hence } \lim \mu_{X_i} = 0.$$

We let $X_\infty = X - X \cdot \bigcup V_i$. Since $X_\infty \subset X_n$ and hence $\mathbf{V}(X_\infty, \alpha) \subset \mathbf{V}(X_n, \alpha)$, for $n = 1, 2, \ldots$, then

$$\mu_{X_\infty} = 0.$$

This means that $\sigma\mathbf{E} = \bigcup V_i \supset X$ (mod $\mathbf{N}^*$), as required, and the proof is complete.

Remarks. If we wish, as does de Possel, to "anchor" the V_n to points of X, we can extract V_n from an x_n-converging sequence, whose constituents belong to $\mathbf{V}(X_n, \alpha)$, and such that $x_n \in X_n$.

As is known **(25**, p. 129**)**, in any Euclidean space the interval basis possesses the density property, therefore, by Theorem 2.12, it is a weak derivation basis. There exists **(17)** an example of a summable function f in the plane whose indefinite integral is I-differentiable (*strongly* differentiable), although the integral of $|f|$ is not.

2.2 A necessary and sufficient condition for a weak derivation basis to differentiate a μ-finite M-measure (Radon measure) ψ. We assume that $\bar{B}$ is a weak derivation basis; that is, $\bar{B}$ possesses the Vitali μ-property.

We let f denote a μ-measurable, non-negative, almost everywhere finite function with domain R; by f_n we shall mean that function for which $f_n(x) = 0$ if $f(x) > n$ and $f_n(x) = f(x)$ if $f(x) < n$. We further define $r_n(x) = f(x) - f_n(x)$, and for $M \in \mathbf{M}$,

$$\psi(M) = \int_M f(x)\,d\mu; \quad \psi_n(M) = \int_M f_n(x)\,d\mu; \quad \rho_n(M) = \int_M r_n(x)\,d\mu.$$

Since f_n is a μ-measurable bounded function, $\bar{B}$ differentiates its integral ψ_n; that is, $D\psi_n$ exists almost everywhere in E and equals f_n(mod $\mathbf{N}^*$). We have

$$D^*\psi = D\psi_n + D^*\rho_n,$$

hence

$$D^*\psi = f_n + D^*\rho_n \text{ (mod } \mathbf{N}^*)$$

on E. In accordance with the definition of f_n and the finiteness (mod $\mathbf{N}^*$), we have $\lim f_n = f$ almost everywhere on E. This leads to the following result.

LEMMA 2.21. *A necessary and sufficient condition for a weak derivation basis to differentiate ψ is*

$$\lim\,[D^*\rho_n > 0] = 0.$$

In particular, if $\bar{\mu}(E)$ is finite, this is equivalent to the condition that $\lim \bar{\mu}[D^*\rho_n > \epsilon] = 0$ *for each positive* ϵ.

COROLLARY 2.22. *If $\bar{B}$ differentiates the non-negative μ-integral ψ, it differentiates any* $\mathbf{M}$*-measure ψ' for which $\psi' < \psi$.*

This result can be extended to any μ-finite (resp., Radon) **M**-measure ψ. In fact, for $M \in \mathbf{M}$,

$$\psi(M) = \psi_s(M) + \int_M f(x)\, d\mu,$$

where f represents the Radon-Nikodym integrand of ψ. We suppose that ψ' is any **M**-measure, $\psi' < \psi$. Then

$$\psi'(M) = \psi'_s(M) + \int_M f'(x)\, d\mu$$

is the corresponding decomposition for ψ'.

If N_0 denotes an **N**-set for which $\psi_s(R - N_0) = 0$, then ψ is μ-absolutely continuous on the **M**-subsets of $R - N_0$, consequently, so is ψ'; therefore $\psi'_s(R - N_0) = 0$,

$$\psi'_r(M) = \psi'(M \cdot (R - N_0)) < \psi(M \cdot (R - N_0)) = \psi_r(M),$$
$$\psi'_s(M) = \psi'(M \cdot N_0) < \psi(M \cdot N_0) = \psi_s(M).$$

Thus ψ'_r and ψ'_s are dominated by ψ_r and ψ_s, respectively. The assumption that $\bar{B}$ differentiates ψ means that $D\psi = f$ (mod $\mathbf{N}^*$) on E, hence $D\psi_r = f$ (mod $\mathbf{N}^*$) on E, and $D\psi_s = 0$ (mod $\mathbf{N}^*$) on E. Since $D^*\psi'_s < D^*\psi_s$, then $D\psi'_s$ exists and equals zero (mod $\mathbf{N}^*$) on E. Thus, we have the following general result.

THEOREM 2.23. *If a weak derivation basis $\bar{B}$ differentiates the μ-finite (resp., Radon)* **M**-*measure ψ, then $\bar{B}$ differentiates any μ-finite (resp., Radon)* **M**-*measure dominated by ψ.*

COROLLARY 2.24. *If a weak derivation basis differentiates the total variation τ of a signed μ-finite (Radon)* **M**-*measure ψ, it differentiates ψ itself.*

As a special case, if the weak basis $\bar{B}$ differentiates the integral $\int |f(x)| d\mu$, where f is a σ-bounded measurable function, then $\bar{B}$ differentiates $\int f(x) d\mu$.

LEMMA 2.25. *If the weak derivation basis $\bar{B}$ differentiates the μ-finite Radon* **M**-*measure ψ, $M \in$* **M**, *and $\tau = \psi + \mu$, then the τ-density*

$$\lim_{\iota} \frac{\tau(M \cdot M_\iota(x))}{\tau(M_\iota(x))}$$

exists almost everywhere on E and equals C_M (characteristic function of M).

Proof. We let f denote the Radon-Nikodym integrand of ψ. Then, for $M' \in \mathbf{M}$,

$$\begin{aligned}\tau_M(M') = \tau(M' \cdot M) &= \psi(M' \cdot M) + \mu(M' \cdot M)\\ &= \int_{M' \cdot M} f(x)\, d\mu + \psi_s(M' \cdot M) + \mu(M' \cdot M)\\ &= \int_{M' \cdot M} (f(x) + 1)\, d\mu + \psi_s(M' \cdot M)\\ &= \int_{M'} C_M(x)(f(x) + 1)\, d\mu + \psi_s(M' \cdot M),\end{aligned}$$

where ψ_s is the μ-singular part of ψ. Since $\bar{\mathcal{B}}$ differentiates τ, $\lim \tau(M_\iota(x))/\mu(M_\iota(x))$ exists and equals $f(x) + 1$ for μ^*-almost all $x \in E$. But $\bar{\mathcal{B}}$ also differentiates τ_M, so that

$$\lim \tau(M_\iota(x) \cdot M)/\mu(M_\iota(x))$$

exists and, by above, is equal to $C_M(x)(f(x) + 1)$ for μ^*-almost all $x \in E$. Hence, by division, $\lim \tau(M_\iota(x) \cdot M)/\tau(M_\iota(x))$ exists and equals C_M for μ^*-almost all $x \in E$.

LEMMA 2.26. *If $\bar{\mathcal{B}}$, ψ, and τ are as in the preceding lemma, X is a subset of E of finite outer measure, M_1 is a measure-cover of X, $\mathbf{V}$ is a $\bar{\mathcal{B}}$-fine covering of X, ϵ is a positive number, and $0 < \alpha < 1$, then there exists a finite or infinite sequence of $\mathbf{V}$-sets V_n for which*

2.261 $$\bigcup_n V_n \supset X (\mathrm{mod}\, \mathbf{N}^*), \quad \sum_n \tau(V_n) < \tau\Big(M \cdot \bigcup_n V_n\Big)/\alpha.$$

Proof. For any $\mathbf{M}$-set such that $\bar{\mu}(M \cdot E) > 0$, we define $\mathbf{V}(\tau, M, \alpha)$, as the family of $\mathbf{V}$-sets V for which

2.262 $$\tau(M \cdot V) > \alpha\tau(V),$$

and $\mu(\tau, M)$ as the supremum of the numbers $\mu(V)$ for $V \in \mathbf{V}$ (τ, M, α). From Lemma 2.25, it follows that there is at least one point $x \in M \cdot E$ at which the τ-density of M equals 1, hence $\mathbf{V}(\tau, M, \alpha)$ is non-vacuous, and $\mu(\tau, M) > 0$. In case $M \in \mathbf{M}$ and $\bar{\mu}(M \cdot E) = 0$, we define $\mu(\tau, M) = 0$.

From this point on the proof follows closely that of Theorem 2.12, with τ replacing μ and the measure covers having to be specially selected, since τ need not be μ-absolutely continuous. By a process similar to that of Theorem 2.12, for fixed κ, $0 < \kappa < 1$, we determine inductively a finite or infinite sequence $V_1, V_2, \ldots$ of $\mathbf{V}$-sets with properties as follows. We put $X_1 = X$, and for any positive integer $n \geqslant 1$,

$$X_{n+1} = X_1 - X_1 \cdot \bigcup_{i=1}^{n} V_i,$$

M_{n+1} denotes a measure cover of X_{n+1} contained in

$$M_n - M_n \cdot \bigcup_{i=1}^{n} V_i.$$

If $\bar{\mu}(X_{n+1}) > 0$, then V_{n+1} is so chosen from $\mathbf{V}$ that

2.263 $$\tau(M_{n+1} V_{n+1}) > \alpha\tau(V_{n+1}), \quad \mu(V_{n+1}) > \kappa\mu(\tau, M_{n+1}).$$

If $\bar{\mu}(X_{n+1}) = 0$, the process stops.

Our choice of the sets M_n ensures that the sets $M_n \cdot V_n$ are strictly disjoint; hence, using 2.263, we have

2.264 $$\tau(M_1) \geqslant \tau\Big(M_1 \cdot \bigcup_n V_n\Big) \geqslant \tau\Big(\bigcup_n M_n \cdot V_n\Big) = \sum_n \tau(M_n \cdot V_n) > \alpha \sum_n \tau(V_n),$$

which is the second relation of 2.261.

If V_n is a finite sequence, then $\bar{\mu}(X_n) = 0$ holds for some integer N, and the first relation of 2.261 clearly holds. If V_n is infinite, we let $X_\infty = X - X \cdot \bigcup V_n$; we may, and do, choose a μ-cover M_∞ of X_∞, contained in $\bigcap M_n$. From 2.263 and 2.264 we have $\lim \mu(\tau, M_n) = 0$. Since $M_\infty \subset M_n$, we have $\mathbf{V}(\tau, M_\infty, \alpha) \subset \mathbf{V}(\tau, M_n, \alpha)$, and $\mu(\tau, M_\infty) < \mu(\tau, M_n)$ for $n = 1, 2, \ldots$; thus $\mu(\tau, M_\infty) = 0$, and $\bar{\mu}(M_\infty \cdot E) = 0$. But $X_\infty \subset M_\infty \cdot E$; hence $\bar{\mu}(X_\infty) = 0$, and the first condition of 2.261 holds.

THEOREM 2.27. *If a weak derivation basis $\bar{\mathfrak{B}}$ differentiates the μ-finite (Radon) $\mathbf{M}$-measure ψ, then $\bar{\mathfrak{B}}$ possesses the Vitali ψ-property.*

Proof. Taking X, M_1, $\mathbf{V}$, and ϵ as in the statement of Lemma 2.26, we select α so that

2.271 $$0 < (\alpha^{-1} - 1)\, \tau(M_1) < \epsilon,$$

and choose an M-family $\mathbf{E}$ in accordance with Lemma 2.26, satisfying 2.261. For $S = \sigma\mathbf{E}$, the (τ, M_1)-redundancy of covering is given by

$$\left\{\sum_n \tau(V_n) - \tau(S)\right\} + \tau(S - S \cdot M_1) = \sum_n \tau(V_n) - \tau(S \cdot M_1),$$

which, by 2.261 and 2.271, is less than ϵ. Thus the τ-overlap of $\mathbf{E}$ and the (τ, M_1)-overflow of $\mathbf{E}$ are less than ϵ, so that the Vitali τ-property holds. Since $\psi < \tau$, the Vitali ψ-property also holds.

Remark. If desired, the sets V_n may be "anchored" to points of X_n, as in the de Possel theorem.

Combining Theorem 2.27 and Theorem 1.52, we obtain the following criterion of differentiability of an individual $\mathbf{M}$-measure.

THEOREM 2.28. *A necessary and sufficient condition for a weak derivation basis $\bar{\mathfrak{B}}$ to differentiate the μ-finite (Radon) $\mathbf{M}$-measure ψ is the validity of the Vitali ψ-property.*

THEOREM 2.29. *The Vitali property for μ-finite (resp., Radon) μ-integrals is equivalent to the $\bar{\mathfrak{B}}$-differentiability of every μ-finite (resp., Radon) μ-integral; the Vitali property for μ-finite (resp., Radon) $\mathbf{M}$-measures is equivalent to the $\bar{\mathfrak{B}}$-differentiability of every μ-finite (resp., Radon) $\mathbf{M}$-measure.*

Proof. This follows from Theorem 1.52, Theorem 2.12, and Theorem 2.27.

2.3. Relation to Younovitch's equivalence theorem. We return to the setting of 1.7 in the following discussion.

Younovitch formulates a Vitali μ-property (Y2) exhibiting the features (i), (ii), and (iii) in the Remarks under 1.71, and asserts its equivalence with the density property. That is, under his assumptions, Younovitch proves Theorem 2.12 with a weakened Vitali μ-property in which X is required to belong to $\mathbf{M}$. We have been unable to prove this; otherwise, Younovitch's theory would be a special instance of ours.

Younovitch also formulates a criterion for the differentiation of μ-integrals, which are necessarily finite since $\mu(R) < \infty$ for his space R. This will be stated after some preliminary definitions are given.

If $\mathfrak{D}$ is a decomposition of the space R into a sequence of disjoint μ-measurable sets $R_1, R_2, \ldots$, so that $R = \bigcup R_\nu$, then $\mathfrak{D}$ is *Y-summable* if and only if

$$\sum_\nu \nu\mu(R_\nu)$$

is finite. For any positive integer k and any positive number ϵ, $U_\mu(\mathfrak{D}, k, \epsilon)$ denotes the set of points x in R for which there exists a sequence

$$A_{n_i}(x)$$

satisfying the relation

$$\sum_{\nu=k}^{\infty} \nu\mu(R_\nu \cdot A_{n_i}(x)) > \epsilon\mu(A_{n_i}(x)).$$

Younovitch's basis $\bar{B}$ is said to have the property (Y3) if and only if for each Y-summable decomposition $\mathfrak{D}$ of R and each positive number ϵ,

$$\lim_k \bar{\mu}\{U_\mu(\mathfrak{D}, k, \epsilon)\} = 0.$$

(Younovitch does not place a bar over μ, evidently regarding the bracketed set as μ-measurable, which seems to confirm the conjecture of 1.71 that he establishes the $\mu(=\mu^*)$-measurability of the derivatives of μ-integrals).

Younovitch's criterion 2.31. (Y2) and (Y3) together are equivalent to the $\bar{B}$-differentiability of every (finite) μ-integral.

Assuming (Y2) to be equivalent to the Vitali μ-property, this result can be proved from the theorems of 2.1, as will now be indicated. Corresponding to the decomposition occurring in the formulation of (Y3), we define a function ϕ by $\phi(x) = \nu$ for $x \in R_\nu, \nu = 1, 2, \ldots$; ϕ is the frequency function of the M-family $\mathbf{C}$ consisting of the sets

$$S_\nu = \bigcup_{i=\nu}^{\infty} R_i.$$

$\mathbf{C}$ is a covering of R. The Y-summability of the decomposition means the finite integrability of $\phi = \phi_{\mathbf{C}}$ over R; conversely, any μ-measurable function on R taking only positive integral values may be regarded as the frequency function of a measurable M-family covering R.

We apply the considerations of 2.2 to $f = \phi$. We have

$$\sum_{\nu=k}^{\infty} \nu\mu(R_\nu \cdot A_{n_i}(x)) = \rho_k(A_{n_i}(x)).$$

Younovitch's set $U_\mu(\mathfrak{D}, k, \epsilon)$ satisfies the relations

$$[D^{*}\rho_k > \epsilon] \supset U_\mu(\mathfrak{D}, k, \epsilon) \supset [D^{*}\rho_k > \epsilon].$$

From Lemma 2.21, assuming the Vitali μ-property, it follows that condition (Y3) is equivalent to $D^*\psi = D_*\psi = f$ (mod $\mathbf{N}^*$) for the integrals of measurable frequency (or multiplicity) functions f.

By Theorem 2.12, the Vitali μ-property is equivalent to the differentiability property when the measurable function f takes only a finite number of values, or even only the values 0 and 1. These latter functions are the characteristic functions of μ-measurable sets.

Combining, we see that Younovitch's theorem amounts to the assertion that a necessary and sufficient condition for the validity of the differentiability of μ-integrals is its validity for the integrals of μ-measurable functions taking only positive integral values.

It remains to be shown only that a weak derivation basis differentiating the integrals of μ-measurable frequency functions differentiates all μ-integrals $\psi(M) = \int_M f(x)\,d\mu$. This follows using the decomposition $\psi(M) = \psi^+(M) + \psi^-(M)$, and for non-negative f, the representation $f(x) = [n](x) + e(x)$, where $[n](x)$ takes only positive integral values and $-1 < e(x) < 0$. Any weak derivation basis differentiates the integrals of the functions e.

2.4. A converse theorem for bases differentiating the $\mu^{(s)}$-functions. In what follows, we assume that the basis $\bar{B}$ is a general derivation basis and that R has the property (G_s).

E denoting an M-family of $\bar{B}$-constituents, r and α positive numbers, we let $\mathbf{E}(\alpha, r)$ denote the family of **E**-sets V for which

$$\int_V \{\epsilon_{\mathbf{E}}(x)\}^r d\mu > \alpha\mu(V),$$

and we further let $\sigma_{\alpha,r}(\mathbf{E})$ denote the union of the sets $\mathbf{E}(\alpha, r)$. Clearly, if $r' > r''$, then $\mathbf{E}(\alpha, r') \supset \mathbf{E}(\alpha, r'')$, and $\sigma_{\alpha,r'}(\mathbf{E}) \supset \sigma_{\alpha,r''}(\mathbf{E})$.

LEMMA 2.41. *If* **H** *represents the* M*-family of the* **E***-constituents* V *for which*

$$\int_V \{\epsilon_{\mathbf{E}}(x)\}^r d\mu < \alpha\mu(V);$$

that is, if $\mathbf{H} = \mathbf{E} - \mathbf{E}(\alpha, r)$, *then*

$$\omega^{(r+1)}(\mathbf{H}) < \alpha \sum_{V \in \mathbf{H}} \mu(V),$$

where $\omega^{(r+1)}(\mathbf{H})$ *denotes the* $\mathfrak{L}^{(r+1)}$*-overlap of* **H**.

Proof.

$$\begin{aligned}\omega^{(r+1)}(\mathbf{H}) &= \int_{\sigma\mathbf{H}} \{\epsilon_{\mathbf{H}}(x)\}^{r+1} d\mu = \int_{\sigma\mathbf{H}} \{\phi_{\mathbf{H}}(x) - 1\}^{r+1} d\mu \\ &< \int_{\sigma\mathbf{H}} \{\phi_{\mathbf{E}}(x) - 1\}^r \phi_{\mathbf{H}}(x)\,d\mu \\ &= \sum_{V \in \mathbf{H}} \int_V \{\epsilon_{\mathbf{E}}(x)\}^r d\mu < \alpha \sum_{V \in \mathbf{H}} \mu(V).\end{aligned}$$

In the preceding considerations, if $r = 0$, we shall interpret $\{\phi_{\mathbf{E}}(x) - 1\}^r$ as the function defined on $\sigma\mathbf{E}$, taking the value 0 if $\epsilon_{\mathbf{E}}(x) = 0$, or the value 1 if $\epsilon_{\mathbf{E}}(x) > 1$; that is, the restriction to $\sigma\mathbf{E}$ of the characteristic function

$C_{\theta E}$ of the **E**-overlap set $\theta\mathbf{E}$ (see Definitions 1.32). $\mathbf{E}(\alpha, 0)$ is the family of the **E**-sets V for which $\mu(V\cdot\theta\mathbf{E}) > \alpha\mu(V)$.

The above lemma remains true when $r = 0$, since

$$\omega^{(1)}(\mathbf{H}) = \omega(\mathbf{H}, \mu) = \int_{\sigma\mathbf{H}} \{\phi_{\mathbf{H}}(x) - 1\}\, d\mu$$
$$< \int_{\sigma\mathbf{H}} C_{\theta\mathbf{H}}(x)\phi_{\mathbf{H}}(x)\, d\mu < \sum_{V\epsilon\mathbf{H}} \int_V C_{\theta\mathbf{E}}(x)\, d\mu < \alpha \sum_{V\epsilon\mathbf{H}} \mu(V).$$

DEFINITION 2.42. We say that the basis $\bar{B}$ has the property (H_p), for $p > 1$, if and only if for any bounded set $X \subset E$, any $\bar{B}$-fine covering **V** of X, any $z^* > \bar{\mu}(X)$, any $\epsilon^* > 0$, and any $\alpha^* > 0$, there exists a finite M-family **G** of **V**-sets such that

2.421 $$\bar{\mu}(X - X \cdot \sigma\mathbf{G}) < \epsilon^*; \ \sum_{V\epsilon\mathbf{G}} \mu(V) < z^*; \ \mu(\sigma_{\alpha^*,\, p-1}(\mathbf{G})) < \epsilon^*.$$

Remarks. Without the third condition, we have the Vitali μ-property in the ϵ-version, and for bounded subsets of E, which, as noted earlier, is equivalent to the original definition of the Vitali μ-property.

We see that if $p' > p''$, then $(H_{p'})$ implies $(H_{p''})$.

LEMMA 2.43. *If $\bar{B}$ is an $S^{(z)}$-basis, $z > 1$, and if $\bar{B}$ does not possess the property $(H_{p'})$, where $p' > 1$, then there exists a bounded set $X_0 \subset E$, a bounded* **G**-*set $G_0 \supset X_0$, a $\bar{B}$-fine covering* $\mathbf{V}_0$ *of X_0, and positive numbers α_0, ϵ_0 such that for every* M-*family* **F** *of* $\mathbf{V}_0$-*sets satisfying the relations*

2.431 $$\bar{\mu}(X_0 - X_0 \cdot \sigma\mathbf{F}) < \epsilon_0, \quad \mu(\sigma\mathbf{F} - \bar{X}_0 \cdot \sigma\mathbf{F}) < \epsilon_0, \quad \omega^{(z)}(\mathbf{F}) < \epsilon, \sigma\mathbf{F} \subset G_0,$$

we have

$$\mu\{\sigma_{\alpha_0,p'-1}(\mathbf{F})\} > 2\epsilon_0.$$

Proof. If G is a bounded **G**-set, $X \subset E\cdot G$, **V** is a $\bar{B}$-fine covering of X, α and ϵ are both positive numbers, then we call the entity $(X, G, \mathbf{V}, \alpha, \epsilon)$ an *admissible quintuple.* Since $\bar{B}$ is an $S^{(z)}$-basis, for any such quintuple there exist M-familes **F** of **V**-sets for which $\bar{\mu}(X - X\cdot\sigma\mathbf{F}) < \epsilon$, $\mu(\sigma\mathbf{F} - \bar{X}\cdot\sigma\mathbf{F}) < \epsilon$, $\omega^{(z)}(\mathbf{F}) < \epsilon$ and $\sigma\mathbf{F} \subset G$. For such families **F** we thus have

$$\sum \mu(V) = \mu(\sigma\mathbf{F}) + \omega(\mathbf{F}, \mu) < \mu(\sigma\mathbf{F}) + \omega^{(z)}(\mathbf{F}) < \bar{\mu}(X) + 2\epsilon.$$

For any fixed admissible quintuple, we let η denote the infimum, among all such families **F**, of the numbers $\mu(\sigma_{\alpha,p'-1}(\mathbf{F}))$. It follows that if, for each admissible quintuple, the corresponding η were zero, then $\bar{B}$ would have the property $(H_{p'})$, contrary to hypothesis. Thus, for some admissible quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$, the corresponding η_0 is a positive number, and for each finite M-family **F** of $\mathbf{V}_0$-sets satisfying the relations 2.431, we have

2.532 $$\mu\{\sigma_{\alpha,p'-1}(\mathbf{F})\} > \eta_0 > 0.$$

Now if **F** is any M-family which satisfies the relations obtained from 2.431 merely by replacing ϵ_0 by any smaller positive number, then **F** necessarily

satisfies the unchanged relation 2.431. Thus, we may assume that ϵ_0 has been chosen so small that $0 < \epsilon_0 < \frac{1}{2}\eta_0$, which, in the light of 2.432, completes the proof.

Henceforth, any quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$ satisfying the conditions of Lemma 2.43 will be called a *privileged quintuple.*

LEMMA 2.44. *If the basis $\bar{\mathfrak{B}}$ possesses the property (H_p), where $p > 1$, then $\bar{\mathfrak{B}}$ is an $S^{(p)}$-basis.*

Proof. By virtue of the Remarks following Theorem 1.67, it suffices to show that the $S^{(p)}$-properties hold when X is any bounded subset of E. Thus, we take a bounded set $X \subset E$, a $\bar{\mathfrak{B}}$-fine covering $\mathbf{V}$ of X, choose $\epsilon > 0$, and select z^* so that $\bar{\mu}(X) < z^* < \bar{\mu}(X) + \frac{1}{2}\epsilon$. We let $\epsilon^* = \frac{1}{4}\epsilon$ and choose any positive α^* with $\alpha^* z^* < \epsilon$.

We use the property (H_p) to find a finite M-family $\mathbf{G}$ of $\mathbf{V}$-constituents such that 2.421 holds. We define $\mathbf{F}$ as the family of $\mathbf{G}$-sets V for which

$$\int_V \{\epsilon_{\mathbf{G}}(x)\}^{p-1} d\mu < \alpha^* \mu(V).$$

If $Y = X \cdot \sigma\mathbf{G}$, then $\mathbf{F}$ covers $Y - Y \cdot \sigma_{\alpha^*, p-1}(\mathbf{G})$. Hence, by 2.421, and the definition of ϵ^*, we have

2.441 $$\bar{\mu}(X - X \cdot \sigma\mathbf{F}) < \bar{\mu}(X - X \cdot \sigma\mathbf{G}) + \mu(\sigma_{\alpha^*, p-1}(\mathbf{G})) < 2\epsilon^* = \epsilon/2;$$

that is, $\mathbf{F}$ is an ϵ-covering of X.

Using Lemma 2.41, with $r = p - 1$, and taking account of the second relation in 2.421 and the choice of α^*, we obtain

$$\omega^{(p)}(\mathbf{F}) < \alpha^* \sum_{V \in \mathbf{F}} \mu(V) < \alpha^* \sum_{V \in \mathbf{G}} \mu(V) < \alpha^* z^* < \epsilon.$$

Finally, from conditions 2.421 we have

$$\begin{aligned}\mu(\sigma\mathbf{F} - \bar{X} \cdot \sigma\mathbf{F}) &< \mu(\sigma\mathbf{G} - \bar{X} \cdot \sigma\mathbf{G}) = \mu(\bar{X} - \bar{X} \cdot \sigma\mathbf{G}) + \mu(\sigma\mathbf{G}) - \mu(\bar{X}) \\ &< \mu(\bar{X} - \bar{X} \cdot \sigma\mathbf{G}) + \Big\{\sum_{V \in \mathbf{G}} \mu(V) - \mu(\bar{X})\Big\} < \epsilon,\end{aligned}$$

which completes the proof that $\bar{\mathfrak{B}}$ is an $S^{(p)}$-basis.

DEFINITION 2.45. We assume R to be a measure space as described in 1.1. We suppose that $\mathbf{U}$ is a given family of $\mathbf{M}$-sets of finite positive measure, δ a positive finite function defined on $\mathbf{U}$. We define E as the set of points x for which there exists at least one ordinary sequence of sets $V_n \in \mathbf{U}$ with $\lim \delta(V_n) = 0$ and $x \in V_n$ for $n = 1, 2, \ldots$. We define the *$\mathfrak{D}$-basis* $[\mathbf{U}, \delta]$ by associating with each $x \in E$ the totality of ordinary sequences of sets $M_\iota(x)$ $(\iota = 1, 2, \ldots)$ for which $x \in M_\iota(x)$, $M_\iota(x) \in \mathbf{U}$, and $\lim \delta(M_\iota(x)) = 0$. From our assumptions it follows that the domain of $[\mathbf{U}, \delta]$ is E and the spread is a subfamily of $\mathbf{U}$. The function δ is called the *index of uniform contraction.*

For any fixed $\eta > 0$, we denote by $\mathbf{V}_\eta$ the subfamily of $\mathbf{V}$ consisting of the $\mathbf{V}$-sets V for which $\delta(V) < \eta$.

For a family (or an M-family) $\mathbf{F}$ of V-sets V, we define the δ-fineness or δ-norm $\nu(\mathbf{F})$ as sup $\delta(V)$, for $V \in \mathbf{V}$.

Remarks. Denjoy **(2)** considered bases more general than those just defined, insofar as the requirement "$x \in M_\iota(x)$ for each ι" is replaced by "$x \in E(M_\iota)$ for each ι," where $E(V)$ is defined for each $\bar{B}$-constituent V as a subset of R, not necessarily μ-measurable, containing V. However, the contraction requirement is lim $\mu(M_\iota) = 0$ **(19)**. Nevertheless, we name the bases introduced here after Denjoy since his memoir points to their specific properties. Following Haupt, they are called "U-Basen" in **(21**, p. 71**)**, for reasons there explained.

Once δ is fixed, a $\mathfrak{D}$-subbasis of $[\mathbf{U}, \delta]$ is uniquely defined by its spread $\mathbf{T} \subset \mathbf{U}$; its domain $D[\mathbf{T}, \delta]$ (abbreviated $D[\mathbf{T}]$) is no longer an arbitrary subset of D[U] as is the case with a general subbasis. For instance, if $\bar{B}$ is an $S^{(1)}$-basis, the domain of any $\mathfrak{D}$-subbasis of $\bar{B}$ is a μ^*-measurable set.

A $\bar{B}$-fine covering $\mathbf{V}$ of X is characterized as a subfamily of $\mathbf{U}$ with $D[\mathbf{V}] \supset X$ (mod $\mathbf{N}^*$).

LEMMA 2.46. *If $\bar{B}$ is a $\mathfrak{D}$-basis* $[\mathbf{U}, \delta]$, $p' > z > 1$, *and $\bar{B}$ is an $S^{(z)}$-basis but not an $S^{(p')}$-basis, then there exists a bounded* $\mathbf{G}$*-set G_0 and a set X_0, with $G_0 \cdot E \supset X_0$, a $\bar{B}$-fine covering $\mathbf{V}_0$ of X_0, positive numbers α_0 and ϵ_0, and a sequence,* $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n, \ldots$ *of finite* M*-families of* $\mathbf{V}_0$*-sets for which*

2.461 $$\lim \nu(\mathbf{F}_n) = 0, \quad S_n = \sigma F_n \subset G_0;$$

2.462 $$\bar{\mu}(X_0 - X_0 \cdot S_n) < \epsilon_0, \quad \omega^{(z)}(\mathbf{F}_n) < \epsilon_0/2^{n+1}, \quad \mu(\sigma_{\alpha_0, p'-1}(\mathbf{F}_n)) > 2\epsilon_0.$$

Proof. By Lemma 2.44, $\bar{B}$ does not possess the property (H_p'), and we may apply Lemma 2.43 t find a privileged quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$. Since $\bar{B}$ is an $S^{(z)}$-basis, we define $\mathbf{F}_n$ as a finite M-family of $\mathbf{V}_{1/n}$-sets included in G_0, satisfying the first two relations in 2.462. The last relation in 2.462 holds by our choice of a privileged quintuple, and 2.461 is clearly valid.

LEMMA 2.47. *We let $\bar{B}$ denote an $S^{(1)}$-basis which is also a $\mathfrak{D}$-basis* $[\mathbf{U}, \delta]$. *We define p_0 as the supremum of numbers p such that $\bar{B}$ is an $S^{(p)}$-basis; we assume that $p_0 < \infty$. We so define q_0 that $p_0^{-1} + q_0^{-1} = 1$ if $p_0 > 1$; otherwise $q_0 = \infty$ if $p_0 = 1$. Then for any number q, $1 < q < q_0$, there exists a $\mu^{(\sigma)}$-function ψ_0, a positive number α_0, and a subset C_0 of E of positive outer measure such that:*

2.471 $$\psi_0(M) = \int_M f_0(x)\, d\mu \text{ for } M \in \mathbf{M}, \text{ and } \int_R |f_0(x)|^q d\mu < \infty;$$

2.472 $$f_0(x) = 0 \text{ for each } x \in C_0;$$

2.473 $$D^* \psi_0 > \alpha_0 > 0 \text{ for each } x \in C_0.$$

Proof. We so define p that $p^{-1} + q^{-1} = 1$. Our hypotheses on q ensure that $p_0 < p < \infty$. In case $p_0 > 1$, we clearly have

$$0 < q(p_0 - 1) < q_0(p_0 - 1) = p_0;$$

hence we can choose a number p' so that

$$0 < q(p' - 1) < p_0 < p' < p.$$

Even in case $p_0 = 1$, this last inequality may be satisfied for a suitable choice of p'.

In either case, we so define q' that $(p')^{-1} + (q')^{-1} = 1$; clearly, then, $q < q' < q_0$. We let z denote the larger of the two numbers $q(p' - 1)$ and 1.

From our assumptions, it follows that $\bar{B}$ is an $S^{(z)}$-basis but not an $S^{(p')}$-basis. Lemma 2.46 asserts the existence of a privileged quintuple $(X_0, G_0, \mathbf{V}_0, \alpha_0, \epsilon_0)$ and a sequence $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n, \ldots$ of finite M-families of $\mathbf{V}_0$-sets satisfying 2.461 and 2.462. We let

$$S_n = \sigma\mathbf{F}_n, O_n = \theta\mathbf{F}_n(\mathbf{F}_n\text{-overlap set}), \quad D = \bigcup_n O_n,$$

$$H_n = \sigma_{\alpha_0, p'-1}(\mathbf{F}_n), Q_n = H_n - H_n \cdot D, \quad C_0 = \limsup Q_n,$$

$$\epsilon_n(x) = \epsilon_{\mathbf{F}_n}(x) \text{ if } x \in O_n, \epsilon_n(x) = 0, x \notin O_n.$$

We have

$$\mu(D) < \sum_n \mu(O_n) < \sum_n \omega(\mathbf{F}_n) < \sum_n \omega^{(z)}(\mathbf{F}_n) < \epsilon_0.$$

Since $\mu(H_n) > 2\,\epsilon_0$ for each n, then $\mu(Q_n) > \epsilon_0$ for all n. Since the sets Q_n are subsets of the set G_0 of finite outer measure, then $\mu(C_0) > \epsilon_0$.

We define

2.474 $$f_0(x) = \sum_n \{\epsilon_n(x)\}^{p'-1}, \quad \psi_0(M) = \int_M f_0(x)\,d\mu, \qquad M \in \mathbf{M}.$$

Now f_0 is non-negative and vanishes on $R - D$, hence on $C_0 \subset (R - D)$; this verifies 2.472.

From the definition of f_0 and Minkowski's inequality, we have

$$\left(\int_R |f_0(x)|^q d\mu\right)^{1/q} < \sum_n \left(\int_R \{\epsilon_n(x)\}^{(p'-1)q} d\mu\right)^{1/q} < \sum_n \left(\int_R \{\epsilon_n(x)\}^z d\mu\right)^{1/q}$$

$$= \sum_n \{\omega^{(z)}(\mathbf{F}_n)\}^{1/q} < \sum_n \left(\frac{\epsilon_0}{2^{n+1}}\right)^{1/q} = \epsilon_0^{1/q} \cdot \sum_n \rho^{n+1},$$

where $\rho = 2^{-1/q}$. Since $q > 1$, then $\rho < 1$, and the sum of the geometrical series is finite; thus 2.471 holds.

We denote by $\mathbf{F}_n(\alpha_0, p' - 1)$ the family of those sets $V \in \mathbf{F}_n$ for which

2.475 $$\int_V \{\epsilon_n(x)\}^{p'-1} d\mu > \alpha_0\mu(V);$$

then $H_n = \sigma\mathbf{F}_n(\alpha_0, p' - 1)$.

To each point $x \in C_0 \subset \limsup H_n$ there corresponds a sequence of natural numbers n_j such that

$$x \in V(n_j),\ V(n_j) \in \mathbf{F}_{n_j}(\alpha_0, p' - 1).$$

The sequence of the sets $V(n_j)$ is an x-contracting sequence of $\bar{B}$, thus $x \in E$, and $C_0 \subset E$.

From 2.474 and 2.475 it follows that

$$\psi_0(V(n_j))/\mu(V(n_j)) > \left(\int_{V(n_j)} \{\epsilon_{n_j}(x)\}^{p'-1} d\mu\right) \Big/ \mu(V(n_j)) > \alpha_0,$$

from which we obtain 2.473.

THEOREM 2.48. *If $\bar{B}$ is a $\mathfrak{D}$-basis which differentiates the $\mu^{(q)}$-functions, where $1 < q < \infty$, and if p is so defined that $p^{-1} + q^{-1} = 1$, then $\bar{B}$ is an $S^{(p')}$-basis for each number p' such that $1 < p' < p$.*

Proof. Since $q > 1$, and $\bar{B}$ differentiates the $\mu^{(q)}$-functions, then $\bar{B}$ must differentiate the $\mu^{(\infty)}$-functions, that is, the integrals of μ-measurable functions which are bounded on each set G°_n. By Theorem 2.12, $\bar{B}$ is an $S^{(1)}$-basis. Next, we define p_0 and q_0 as in Lemma 2.47. In case $p_0 = \infty$, it is clear that $\bar{B}$ is an $S^{(p')}$-basis and the theorem holds. In case $1 \leqslant p_0 < \infty$, Lemma 2.47 tells us that for each number q' such that $1 < q' < q_0$, there exists at least one $\mu^{(q')}$-function which $\bar{B}$ fails to differentiate. Our hypotheses thus compel us to conclude that $q \geqslant q_0$, hence $p \leqslant p_0$, from which the statement of the theorem is seen to be true.

Remarks. Theorem 2.48 is not a clear-cut converse theorem because it does not say that $\bar{B}$ is an $S^{(p)}$-basis. We conjecture that a $\mathfrak{D}$-basis can be constructed, which is an $S^{(p')}$-basis for each $p' < p$, yet fails to be an $S^{(p)}$-basis.

According to Zygmund (**29**, pp. 143–144), the interval basis in the plane differentiates the $\mu^{(q)}$-functions for $q > 1$. Therefore, by Theorem 2.48, it is an $S^{(p)}$-basis for all finite $p \geqslant 1$. There exists a blanket (**10**, pp. 294–295) which is an $S^{(1)}$-basis but is not an $S^{(p)}$-basis for any $p > 1$. The two bases just mentioned are extreme cases in the continuous chain of $S^{(p)}$-bases.

2.5 An individual differentiability criterion of Busemann-Feller type. Throughout this section we assume that $\bar{B}$ is a $\mathfrak{D}$-basis and that (G_σ) holds.

DEFINITION 2.51. For $M \in \mathbf{M}$, we denote by $\dot{M}$ the element (soma) of $\mathfrak{M} = \mathbf{M}/\mathbf{N}$ corresponding to M; that is, the M-coset of $\mathbf{N}$. If $\mathbf{Z}$ is any subfamily of $\mathbf{M}$, and $\mathfrak{Z}$ is its image by mapping $M \to \dot{M}$, then we define the union (mod N) of the $\mathbf{Z}$-sets, namely $\sigma^*(\mathbf{Z})$, as any set corresponding to the union or join of the $\mathfrak{Z}$-elements in $\mathbf{M}/\mathbf{N}$, regarded as a complete lattice (**26**, pp. 378–380). The lattice relation is so defined that $\dot{Z}' > \dot{Z}''$ holds if and only if for

every $Z' \in \dot{Z}$, and $Z'' \in \dot{Z}''$, the relation $Z' \supset Z''$ (mod $\mathbf{N}$) holds. The union $\sigma^*(\mathbf{Z})$, defined modulo $\mathbf{N}$, is thus characterized by the property that every $\mathbf{M}$-set which includes each $\mathbf{Z}$-set (mod $\mathbf{N}$) includes $\sigma^*(\mathbf{Z})$ (mod $\mathbf{N}$). There exist enumerably many $\mathbf{Z}$-sets $Z_1, Z_2, \ldots, Z_n, \ldots$ such that

$$\bigcup_n Z_n = \sigma^*(\mathbf{Z}) \pmod{\mathbf{N}}.$$

If all the $\mathbf{Z}$-sets are included (mod $\mathbf{N}$) in an $\mathbf{M}$-set of finite measure, then $\sigma^*(\mathbf{Z})$ is any $\mathbf{M}$-set of minimal measure including each $\mathbf{Z}$-set (mod $\mathbf{N}$).

Definition 2.52. If M denotes a bounded $\mathbf{M}$-set (nucleus), η a positive number, α a number such that $0 < \alpha < 1$, then, similarly to Busemann and Feller (**1**, p. 230), we define the *halo* $\sigma_{\alpha,\eta}(M)$ as the union (mod $\mathbf{N}$) of the $\bar{B}$-constituents V for which $\mu(M \cdot V) > \alpha\mu(V)$ and $\delta(V) < \eta$.

Definition 2.53. If ψ is an $\mathbf{M}$-measure, $M \in \mathbf{M}$, $\alpha > 0$, and $\eta > 0$, then we denote by $\sigma_{\alpha,\psi,\eta}(M)$ the union (mod $\mathbf{N}$) of the constituents V for which $\psi(M \cdot V) > \alpha\mu(V)$ and $\delta(V) < \eta$. The set $\sigma_{\alpha,\psi,\eta}(M)$ is called a *ψ-halo*.

Remarks. The modification of the Busemann-Feller definition involving the strict union is due to the possible non-measurability of the strict union in our more general sense.

The term "halo" was first used by K. O. Househam in his talks in Capetown, 1950, on A. P. Morse's differentiation theory, to denote Morse's set $\Delta:\beta$ (**14**, p. 208). We diverge from the colloquial use of the term by permitting our halos to have points in common with the nucleus or even to include the nucleus. However, all our halo conditions control the proper halo; that is, the part of the halo outside the nucleus, thus retaining the basic meaning of the term.

The relation between $\sigma_{\alpha,r}(\mathbf{E})$, defined in 2.4, and the notion just defined may be written $\sigma_{\alpha,r}(\mathbf{E}) = \sigma_{\alpha,\psi,\eta}(M)$, provided $M = \theta(\mathbf{E})$, ψ is the indefinite integral of that function coinciding with $\{\epsilon_{\mathbf{E}}\}^r$ on $\theta\mathbf{E}$, zero elsewhere, and $\eta = \nu(\mathbf{E})$.

Theorem 2.53. *If $\bar{B}$ is a $\mathfrak{D}$-basis and an $S^{(1)}$-basis, and ψ is a non-negative μ-integral, then a necessary and sufficient condition that $\bar{B}$ differentiate ψ is the following* halo evanescence condition: *For any bounded non-increasing sequence of* $\mathbf{M}$*-sets M_n, with* $\lim \mu(M_n) = 0$, *any non-increasing sequence η_n of positive numbers with* $\lim \eta_n = 0$, *and any $\alpha > 0$, we have*

$$\mu\{\lim_n (\sigma_{\alpha,\psi,\eta_n}(M_n))\} = 0.$$

Proof. We first establish the sufficiency. We show that the halo evanescence property implies the Vitali ψ-property in the ϵ-covering version. We let X denote a bounded subset of E, $\mathbf{V}$ a $\bar{B}$-fine covering of X, ϵ a positive number. We shall prove the existence of an M-family $\mathbf{E}$ of $\mathbf{V}$-sets which is an ϵ-covering of X, with ψ-overflow with respect to X, and ψ-overlap, both less than ϵ.

For a suitable N, we have $X \subset G^\circ_N$ (recall Definitions 1.31). Pruning **V** if necessary, we may assume that all the **V**-sets lie in G°_N. Since $\bar{B}$ is an $S^{(1)}$-basis and a $\mathfrak{D}$-basis, then corresponding to each positive integer n, there exists an M-family $\mathbf{E}_n$ of **V**-sets such that if $S_n = \sigma\mathbf{E}_n$ then

2.531 $S_n \supset X (\text{mod } \mathbf{N}^*)$, $\mu(S_n - S_n \cdot \bar{X}) < 2^{-n-1}$, $\omega(\mathbf{E}_n, \mu) < 2^{-n-1}$, $\nu(\mathbf{E}_n) < 1/n$.

Putting

$$O_n = \theta\mathbf{E}_n,\ D_n = \bigcup_{k=n}^{\infty} O_k,$$

we have $\mu(O_n) < 1/2^{n+1}$, $\mu(D_n) < 1/2^n$.

Next we define $\alpha = \epsilon/(\bar{\mu}(X) + 1)$. We denote by $\mathbf{H}_n$ the family of those $\mathbf{E}_n$-sets V for which

2.532 $$\psi(V \cdot O_n) < \alpha\mu(V).$$

We have

2.533 $$\omega(\mathbf{H}_n, \psi) = \int_{\sigma\mathbf{H}_n} \epsilon_{\mathbf{H}_n}(x)\, d\psi < \int_{O_n} \phi_{\mathbf{H}_n}(x)\, d\psi$$
$$= \sum_{V \in \mathbf{H}_n} \psi(V \cdot O_n) < \alpha \sum_{V \in \mathbf{H}_n} \mu(V) < \alpha(\mu(S_n) + \omega(\mathbf{E}_n, \mu))$$
$$< \alpha(\bar{\mu}(X) + 2^{-n-1} + 2^{-n-1}) < \alpha(\bar{\mu}(X) + 1) = \epsilon.$$

Now $\mathbf{E}_n$ is an 0-covering of X. The constituents in $\mathbf{E}_n - \mathbf{H}_n$ are included in $\sigma_{\alpha,\psi,1/n}(O_n)$, (mod $\mathbf{N}^*$); therefore H_n covers

$$X - X \cdot \sigma_{\alpha,\psi,1/n}(O_n) (\text{mod } \mathbf{N}^*),$$

and consequently

2.534 $$\sigma\mathbf{H}_n \supset \{X - X \cdot \sigma_{\alpha,\psi,1/n}(D_n)\} (\text{mod } \mathbf{N}^*).$$

Since the sets D_n are all included in the set G°_N of finite outer measure, and form a non-increasing sequence with $\lim \mu(D_n) = 0$, we may invoke the halo evanescence property to conclude that

$$\lim_n \mu(\sigma_{\alpha,\psi,1/n}(D_n)) = 0.$$

There exists a positive number η such that for any **M**-set $M \subset G^\circ_N$, for which $\mu(M) < \eta$, we have $\psi(M) < \epsilon$. We fix n so that

2.535 $$\mu(\sigma_{\alpha,\psi,1/n}(D_n)) < \eta,\ 2^{-n-1} < \eta.$$

We may and do further assume that $\eta < \epsilon$. Then the family $\mathbf{E} = \mathbf{H}_n$ corresponding to this index n satisfies the ϵ-covering condition due to 2.534 and 2.535. Using the second relation in 2.531 and the fact that $\sigma\mathbf{E} \subset S_n \subset G^\circ_N$, we have $\psi(\sigma\mathbf{E} - \sigma\mathbf{E} \cdot \bar{X}) < \epsilon$; that is, the ψ-overflow of **E** with respect to X is less than ϵ. Finally, the ψ-overlap of **E** is less than ϵ by virtue of 2.533.

We now attend to the proof of the necessity. We consider an arbitrary non-increasing sequence of bounded **M**-sets $M_1, M_2, \ldots, M_n \ldots$ with lim

$\mu(M_n) = 0$, an arbitrary non-increasing positive sequence $\delta_1, \delta_2, \ldots, \delta_n \ldots$, with $\lim \delta_n = 0$, and an arbitrary $\alpha > 0$. We put

$$H_n = \sigma_{\alpha,\psi,\delta_n}(M_n),\ H = \bigcap_n H_n.$$

Since the halo $\sigma_{\alpha,\psi,\delta}(M)$ is a non-decreasing function of δ, it follows readily that for any pair of positive integers n, ν, we have

$$H \subset \sigma_{\alpha,\psi,\delta_n}(M_\nu);$$

hence, for each such ν,

2.536 $$H \subset \bigcap_n \sigma_{\alpha,\psi,\delta_n}(M_\nu).$$

For each such pair of positive integers n, ν, there exist enumerably many constituents

$$V^1_{n,\nu}, V^2_{n,\nu}, \ldots, V^j_{n,\nu}, \ldots$$

such that

2.537 $$\delta(V^j_{n,\nu}) < \delta_n, \quad \psi(V^j_{n,\nu} \cdot M_\nu) > \alpha\mu(V^j_{n,\nu}),$$
$$S_{n,\nu} = \bigcup_j V^j_{n,\nu} = \sigma_{\alpha,\psi,\delta_n}(M_\nu) \ (\text{mod } \mathbf{N}).$$

Corresponding to each point x of the set

$$H^*_\nu = \bigcap_n S_{n,\nu},$$

there exists a sequence of $\bar{B}$-constituents W_n satisfying the relations

2.538 $$x \in W_n, \quad \delta(W_n) < \delta_n, \quad \psi(W_n \cdot M_\nu) > \alpha\mu(W_n).$$

We let f denote the integrand of ψ, so define r that

$$r_\nu(x) = f(x) \text{ if } x \in M_\nu, \quad r_\nu(x) = 0 \text{ if } x \notin M_\nu,$$

and let

$$\rho_\nu(M) = \int_M r_\nu(x)\, d\mu,$$

for $M \in \mathbf{M}$. We deduce from 2.538 that

2.539 $$D^*\rho_\nu(x) \geq \alpha$$

for each $x \in H^*_\nu$.

Since $\bar{B}$ is an $S^{(1)}$-basis and differentiates ψ, then by Theorem 2.23, $\bar{B}$ differentiates the non-negative Radon integrals dominated by ψ for almost all $x \in E$. Hence, from 2.539, $r_\nu(x) \geq \alpha$ almost everywhere in H^*_ν. But $r_\nu(x) = 0$ for each $x \in M_\nu$, thus $H^*_\nu \subset M_\nu$ (mod $\mathbf{N}^*$), which, by 2.537, means that

$$\bigcap_n \sigma_{\alpha,\psi,\delta_n}(M_\nu) \subset M_\nu (\text{mod } \mathbf{N}^*);$$

using 2.536, we observe that $H \subset M_\nu$ (mod $\mathbf{N}^*$). Since ν is arbitrary and $\lim \mu(M_\nu) = 0$, we conclude that $\mu(H) = 0$, which completes the proof.

§3. Examples of Bases Possessing the Vitali Property for Radon Measures

Throughout this section the setting of 1.1 is adopted.

3.1 Preliminary definitions.

DEFINITION 3.11. By *external $\mathfrak{D}$-closed* set we shall mean the R-complement of a **G**-set; **A** will denote the family of all such sets.

DEFINITION 3.12. We say that Δ is a *(Morse) disentanglement function* if and only if Δ is a positive finite function defined on the spread (family of the constituents) of the basis $\bar{B}$ (**14**, p. 207).

DEFINITION 3.13. If α is a fixed number greater than 1, Δ is a disentanglement function, and V_0 is a $\bar{B}$-constituent, then the *Morse halo* $H(\Delta, \alpha, V_0)$ is the union of those constituents V which intersect V_0, and for which $\Delta(V) < \alpha\Delta(V_0)$. The *halo dilatation* $\rho(\Delta, \alpha, V_0)$ is defined as the ratio $\bar{\mu}(H(\Delta, \alpha, V_0))/\mu(V_0)$.

DEFINITION 3.14. We shall say that the basis $\bar{B}$ has the *strong Vitali property* (abbreviated (S.V.)) if and only if for each $\epsilon > 0$, each set $X \subset E$ of finite outer measure, and each $\bar{B}$-fine covering **V** of X there exists an (enumerable) M-family **E** of **V**-constituents such that, for $S = \sigma\mathbf{E}$:

(S.V.1) $X - X \cdot S \in \mathbf{N}^*$;
(S.V.2) $\mu(S - S \cdot \tilde{X}) < \epsilon$;
(S.V.3 str.) the **E**-constituents are pairwise disjointed.

If, in (S.V.3 str.) we replace the strict disjunction by 0-disjunction, that is, disjunction mod $\mathbf{N}^*$, we obtain the *strong Vitali property mod* $\mathbf{N}^*$; if we discard (S.V.2), we have the *reduced strong Vitali property* (abbreviated R.S.V.). Recalling the Definitions 1.65, we find it convenient to designate as an $S^{(\infty)}$-basis, any basis having the property (S.V.) mod $\mathbf{N}^*$.

The straightforward proofs of the following are omitted.

PROPOSITION 3.15. (S.V.) mod $\mathbf{N}^*$ *implies the Vitali property for μ-finite μ-integrals.*

PROPOSITION 3.16. (R.S.V.) *and Haupt's adaptation property together imply the Vitali property for Radon measures.*

DEFINITION 3.17. We say that $\bar{B}$ has the *generalized Morse halo property* (**14**, p. 213, Def. 6.4) if and only if there exists $\alpha > 1$ and a disentanglement function Δ for which

$$\sup\{\limsup[\Delta(M_\iota(x)) + \rho(\Delta, \alpha, M_\iota(x))]\} < \infty$$

for μ-almost all points $x \in E$. Here, as in 1.1, the limit superior is taken for a sequence $M_\iota(x)$ and the supremum is taken over all x-converging sequences.

Remarks. The strong Vitali properties lack the flexibility of the Vitali properties of §1. In their formulation, one cannot replace the 0-covering condition by an ϵ-covering one, nor in the (G_σ) case, replace the phrase "of finite outer measure" by "bounded." However, such alterations are permissible if the constituents are **A**-sets.

3.2 Generalized Morse bases.

FUNDAMENTAL THEOREM 3.21. *We suppose that $\bar{B}$ is such a basis that for each $x \in E$, the sets of every x-converging sequence contain x, and each $\bar{B}$-constituent is a member of* **A**. *We assume, further, that each* **M**-*set of finite measure is a subset of some* **G**-*set of finite outer measure, and that Morse's halo property holds. Then $\bar{B}$ possesses the property* (R.S.V.).

The proof of this theorem occurs in (**21**, p. 80).

Remarks. In Morse's version of the fundamental theorem, R is a metric space, μ a classical Radon measure, $\bar{B}$ is a blanket F. Since the contraction process is defined by means of the metric, the (metrically) open sets belong to **G**, the (metrically) closed to **A**. Thus a Radon measure in the classical sense is a Radon measure in the sense of 1.31, the reference sequence $G^\circ_1, G^\circ_2, \ldots$ consisting of concentric open spheres, whose radii tend to infinity. Morse assumes that $V \in F(x)$ implies $x \in V$, and that the $\bar{B}$-constituents are closed. Without the closeness assumption for the constituents, $\bar{B}$ need not be strong as the following examples confirm.

EXAMPLE 3.22. R is a plane Euclidean space, μ is plane Borel measure, and E is the open unit square with principal vertices at $(0, 0)$ and $(1, 1)$. To avoid repetition, throughout this discussion, t will denote an arbitrary point of E, n an arbitrary positive integer. We let $\mathbf{T}_n$ denote the set of points in R of the form $(r/2^n, s/2^n)$, where r and s are arbitrary integers. $\mathbf{K}_n$ denotes the family of closed squares whose four vertices are points of $\mathbf{T}_n$, with sides of length 2^{-n}. Each point t lies in or on the boundary of at least one square in $\mathbf{K}_n$; we associate, with each such t, exactly one square $I_{n,t}$ in $\mathbf{K}_n$ such that $t \in I_{n,t}$. We define $I'_{n,t}$ as the square concentric with $I_{n,t}$, with sides parallel to the axes and three times as long as those of $I_{n,t}$.

At each point $z \in \mathbf{T}_n$, we construct a square centered at z, with sides parallel to the axes, and of length 2^{-2n}. We let $\mathbf{H}_n$ denote the family of all such squares, and we define

$$J_n = \bigcup_{m=n+1}^{\infty} \sigma\mathbf{H}_m.$$

We further define $I''_{n,t} = I_{n,t} + I'_{n,t} \cdot J_n$. Finally, we so define the blanket F with domain E that $F(t)$ is the family consisting of the sets $I''_{1,t}, I''_{2,t}, \ldots, I''_{n,t}, \ldots$.

For each integer $m > n + 1$, there are not over $16 \cdot 2^{2m-2n}$ points of $\mathbf{T}_m$

lying on or in $I_{n,t}$, thus not over $16 \cdot 2^{2m-2n}$ members of $\mathbf{H}_m$, each of μ-measure 2^{-4m}, intersecting $I''_{n,t}$. Therefore

$$\mu(I'_{n,t} \cdot J_n) < 16 \sum_{m=n+1}^{\infty} 2^{-2m} \cdot 2^{-2n} < 2^{-2n+2}\mu(I_{n,t});$$

since $n > 1$, we have

3.221 $$\mu(I''_{n,t}) < \mu(I_{n,t}) + \mu(I'_{n,t} \cdot J_n) < 3\mu(I_{n,t}) < \tfrac{1}{3}\mu(I'_{n,t}).$$

We consider a subblanket of F, say F_1, with spread $\mathbf{F}_1$, having the property that each member of $\mathbf{F}_1$ is included in E. We let $\mathbf{G}$ be any countable subfamily of $\mathbf{F}_1$ whose μ-overlap is zero. Each set $\beta \in \mathbf{G}$ is a set $I''_{n,t}$, and we may associate with β the corresponding set $\beta' = I'_{n,t}$; we let $\mathbf{G}'$ denote the family of the corresponding sets β'. Due to our construction, it follows that $\mathbf{G}'$ has μ-overlap zero. Using 3.221 above and the fact that $\sigma\mathbf{G} \subset E$ we obtain

$$\mu(S - \sigma\mathbf{G}) > \mu(S) - \sum_{\beta \in \mathbf{G}} \mu(\beta) > 1 - \tfrac{1}{3}\sum_{\beta' \in \mathbf{G}'} \mu(\beta') = 1 - \tfrac{1}{3}\mu(\sigma\mathbf{G}') > 1 - \tfrac{2}{3} = \tfrac{1}{3}.$$

Thus, no countable subfamily of $\mathbf{F}_1$ whose μ-overlap is zero can cover μ-almost all of E. At the same time, if we define $\Delta(\beta) = \operatorname{diam} \beta$ for $\beta \in \mathbf{F}$, then it is clear that Morse's halo property holds.

EXAMPLE 3.23. In this example, R and E are both the set of all real numbers, μ is linear Borel measure, V° denotes a fixed open subset of the open interval $I = (-1, 1)$, containing the point $x = 0$, everywhere dense in I, with $\mu(V^\circ) = 2\theta^\circ$, where $0 < \theta^\circ < 1$. We define $V(x, \zeta)$, for $\zeta > 0$, as the open set image of V° by the direct homothetic transformation carrying the interval $(x - \zeta, x + \zeta)$ onto I. $F(x)$ is defined as the family of all sets $V(x, \zeta)$, $\zeta > 0$. We define $\Delta(V) = \mu(V)$ for $V = V(x, \zeta)$; hence $\Delta(V) = 2\zeta\theta^\circ$. From this it follows that $H(\Delta, \alpha, V(\alpha, \zeta)) = (x - \zeta(2\alpha + 1),\ x + \zeta(2\alpha + 1))$; thus $\rho(\Delta, \alpha, V) = (2\alpha + 1)/\theta^\circ$. Since

$$\limsup_{F(x) \ni V \to x} \Delta(V) = 0$$

for each real number x, then Morse's halo property is valid.

We let F_1 denote a subblanket of F with domain I, with the property that the closure of every member of the spread of F_1 is included in I. We consider any countable subfamily $\mathbf{G}$ of the spread of F_1, whose μ-overlap is zero. If $\beta = V(x, \zeta) \in \mathbf{G}$, then the closure β' of β is the closed interval $[x - \zeta, x + \zeta]$, and $\mu(\beta) = \theta^\circ \mu(\beta')$. If $\mathbf{G}'$ denotes the family of the corresponding sets β', it follows from the density of β in β' that the μ-overlap of $\mathbf{G}'$ is zero. Thus, since $\sigma\mathbf{G} \subset \sigma\mathbf{G}' \subset I$, we have

$$\mu(I - \sigma\mathbf{G}) > \mu(I) - \sum_{\beta \in \mathbf{G}} \mu(\beta) = 2 - \theta^\circ \cdot \sum_{\beta' \in \mathbf{G}'} \mu(\beta') = 2 - \theta^\circ \mu(\sigma\mathbf{G}') > 2 - 2\theta^\circ > 0.$$

This shows that F has not the property (S.V.), although F is regular.

THEOREM 3.24. *If $\bar{B}$ is such a basis that each $\bar{B}$-constituent of any x-convergent sequence at any point x of E includes an* $\mathbf{A}$*-set containing x, and if Haupt's*

adaptation property and Morse's halo property both hold, then $\bar{B}$ possesses the Vitali property for Radon measures.

Proof. We let ψ denote an arbitrary Radon measure, X an arbitrary bounded subset of E, say $X \subset G^\circ{}_N$, $\mathbf{V}$ any $\bar{B}$-fine covering of X, and ϵ any positive number. Using Proposition 1.38 we have only to show the existence of an M-family $\mathbf{E}$ of $\mathbf{V}$-sets such that

$$\bar{\mu}(X - X \cdot \sigma\mathbf{E}) < \epsilon, \quad \omega(\mathbf{E}, \psi) < \epsilon.$$

For $n = 1, 2, \dots$ we denote by $\bar{B}^*{}_n$ the set of the $\bar{B}$-sequences $M_\iota(x)$ such that $x \in X$, whose constituents V belong to $\mathbf{V}$, are included in $G^\circ{}_N$, satisfying

3.241 $$\Delta(V) + \rho(\Delta, \alpha, V) < n,$$

by η a number for which $0 < \eta < 1$, and by X_n the domain of $\bar{B}^*{}_n$. We have $X_1 \subset X_2 \subset \dots \subset X_n \dots$. Since $G^\circ{}_N$ is a $\mathbf{G}$-set, $\mathbf{V}$ a $\bar{B}$-fine covering of X, and Morse's halo property holds, it follows that $\lim X_n = X \pmod{\mathbf{N}^*}$, hence $\lim \mu(\bar{X}_n) = \mu(\bar{X})$. Since $\mu(\bar{X})$ is finite, we may and do choose n so that

3.242 $$\mu(\bar{X}) - \mu(\bar{X}_n) < \epsilon.$$

We put $\psi^\circ = \sup \psi(M)$ for $\mathbf{M} \ni M \subset G^\circ{}_N$, and, since $\psi^\circ < \infty$, select δ so that

3.243 $$0 < (\delta^{-1} - 1)\, \psi^\circ < \epsilon, \; 0 < \delta < 1.$$

For each $\bar{B}^*{}_n$-sequence $M_\iota(x)$, we determine all possible sequences $A_\iota(x)$ of $\mathbf{A}$-sets, for which the properties

$$x \in A_\iota(x), \mu(A_\iota(x))/\mu(M_\iota(x)) > \eta, \; \psi(A_\iota(x)) > \delta\psi(M_\iota(x))$$

all hold. The corresponding sequences $A_\iota(x)$ may all be regarded as converging to x. The fact that for *each* $\bar{B}^*{}_n$-sequence such associated sequences exist follows from the first assumption of the theorem and the universal lower approximation property of the $\mathbf{A}$-sets, applied to $\psi + \mu$, which is implied by Haupt's adaptation property (recall Proposition 1.37). Thus if $\bar{A}^*$ denotes the family of the sequences $A_\iota(x)$, then $\bar{A}^*$ is a basis with domain X_n.

Following Morse, we associate with each $\bar{A}^*$-constituent V^* a $\bar{B}^*{}_n$-constituent $V = \mathrm{D}(V^*)$ (the *dilatation* of V^*) satisfying the conditions

$$V^* \subset V, \mu(V^*) > \eta\mu(V), \; \psi(V^*) > \delta\psi(V),$$

and we define on the spread of $\bar{A}^*$ the disentanglement function Δ^* by

3.244 $$\Delta^*(V^*) = \Delta(V),$$

where $V = \mathrm{D}(V^*)$.

We observe that the halo $H^*(\Delta^*, \alpha, V^*)$, which is a union of $\bar{A}^*$-constituents, is included in $H(\Delta, \alpha, V)$, hence

3.245 $$\bar{\mu}(H^*(\Delta^*, \alpha, V^*))/\mu(V^*) = \rho^*(\Delta^*, \alpha, V^*) < \rho(\Delta, \alpha, V)/\eta.$$

Combining 3.241, 3.244, and 3.245, we deduce

3.246 $$\Delta^*(V^*) + \rho^*(\Delta^*, \alpha, V^*) < (\Delta(V) + \rho(\Delta, \alpha, V))/\eta < n/\eta,$$

where $V = D(V^*)$. Thus $\bar{A}^*$ possesses the Morse halo property, in fact, uniformly. The assumptions of Theorem 3.21 hold, hence $\bar{A}^*$ has the property R.S.V. Thus there exists a disjointed M-family $\mathbf{E}^*$ of $\bar{A}^*$-constituents, such that for $S^* = \sigma\mathbf{E}^*$, $S^* \supset X_n \pmod{\mathbf{N}^*}$.

We define $\mathbf{E}$ as the M-family obtained from $\mathbf{E}^*$ by the correspondence $V = D(V^*)$. The $\mathbf{E}$-constituents belong to $\mathbf{V}$ and lie in G°_N. Since $D(V^*) \supset V^*$, then

$$S = \sigma\mathbf{E} \supset S^* \supset X_n \pmod{\mathbf{N}^*}, \quad S \supset \bar{X}_n \pmod{\mathbf{N}}.$$

Using 3.242, we have

$$\mu(\bar{X} - \bar{X}\cdot S) < \mu(\bar{X}) - \mu(\bar{X}_n) < \epsilon.$$

Also, since $S \subset G^\circ_N$, we have $\psi(S) < \psi^\circ < \infty$; hence, from 3.243,

$$\omega(\mathbf{E}, \psi) = \sum_{V \epsilon \mathbf{E}} \psi(V) - \psi(S)$$
$$< \delta^{-1}\Big(\sum_{V^* \epsilon \mathbf{E}^*} \psi(V^*)\Big) - \psi(S^*) = (\delta^{-1} - 1)\,\psi(S^*) < \epsilon.$$

The M-family $\mathbf{E}$ fulfils the required conditions, which means that the basis $\bar{B}$ possesses the Vitali property for Radon measures. The following is the immediate consequence of Theorem 1.64.

COROLLARY 3.25. *Under the assumptions of Theorem* 3.24, $\bar{B}$ *differentiates every Radon measure.*

Remarks. The essential steps in the foregoing proof are (i) the contraction of the $\bar{B}^*_n$-sets into $\mathbf{A}$-sets of nearly equal ψ-measure, with μ-exhaustion power greater than η, (ii) the transfer, expressed by $\Delta^*(V^*) = \Delta(V)$, of the function Δ from the original sets to the new ones. The second step shows the power of Morse's methods residing here in the choice of the new disentanglement function.

In Morse's paper **(14)** it is remarked that the metric axiom

$$\delta(p', p'') = 0 \text{ implies } p' = p''$$

is never used. Discarding this axiom, the (metric) closure P of the set $\{p\}$, consisting of the point p only, is the set P of points x with $\delta(p, x) = 0$. To the various points x of P may be attached different families $F(x)$, but any Borel set containing p must contain P. This is why the first assumption in Theorem 3.24 is satisfied under Morse's relaxed hypotheses.

Morse's halo property in the general case involves the contracting process; this is not so, however, in the special case of uniformity. If, for a blanket F, there exists $\alpha > 1$ and Δ such that $\Delta(V) + \rho(\Delta, \alpha, V)$ is bounded on the spread of F, then the same is true of any blanket with the same spread, in

particular for the blanket F^* so defined that $F^*(x)$ is the family of F-constituents containing x, which is a $\mathfrak{D}$-basis. Not only does F differentiate any Radon measure but F^* does, too.

Referring to Theorem 2.29 and Theorem 3.24, the Examples 3.22 and 3.23 give a negative answer to a question raised by O. Nikodym in 1947: "Is the property (SV) mod $\mathbf{N}^*$ equivalent to the validity of the differentiation theorem for μ-finite μ-integrals?"

FUNDAMENTAL THEOREM 3.26. *We suppose that $\bar{\mathfrak{B}}$ is a basis whose constituents are* **A**-*sets, and that there exists a disentanglement function* Δ *such that*

$$\sup\{\limsup \rho(\Delta, 1, V)\} < \infty$$

almost everywhere on E. We assume that $\bar{\mathfrak{B}}$ has the property (L), *namely, corresponding to any subset X of E of finite outer measure, any $\bar{\mathfrak{B}}$-fine covering* **V** *of X, and any* $\epsilon > 0$, *there exists an* M-*family* **E** *of* **V**-*constituents such that for* $S = \sigma\mathbf{E}$ *we have*

(V1) $\qquad S \supset X \pmod{\mathbf{N}^*},$

(V2) $\qquad \mu(S - S\cdot\tilde{X}) < \epsilon.$

Then $\bar{\mathfrak{B}}$ has the property (S.V.).

Proof. This can be found in (**21**, p. 83). See also the Remarks after Theorem 3.27.

Remarks. Property (L) asserts the existence of an M-family covering $X \pmod{\mathbf{N}^*}$ without any overlap condition. The essential step in the proof of the theorem is called *disentanglement*, and rests upon the halo property. For this reason (L) may be regarded as a rough or pre-Vitali property. In the formulation of (L), (V1) may be replaced by $\mu(\tilde{X} - S\cdot\tilde{X}) < \epsilon$, and in case (UG) holds, (V2) may be dropped.

Property (L) is named after Lindelöf. In fact, if R is a metric space with a countable basis (*separable*, in Fréchet's terminology), μ a Radon measure, F a blanket such that the constituents of $F(x)$ are open sets containing x, then Lindelöf's classical topological property implies (even expresses) the property (L).

In Theorem 3.26, it is not assumed that the constituents of an x-converging sequence contain x, as is the case in Theorem 3.24.

THEOREM 3.27. *If there exists a disentanglement function* Δ *such that*

$$\sup\{\limsup \rho(\Delta, 1, V)\} < \infty$$

almost everywhere on E, and if Haupt's adaptation property and (L) *holds, then $\bar{\mathfrak{B}}$ possesses the Vitali property for Radon measures.*

Proof. We let ψ denote a Radon measure, X a bounded subset of E (we assume $X \subset G^\circ{}_N$), **V** a $\bar{\mathfrak{B}}$-fine covering of X, ϵ a positive number. Due to

Proposition 1.38, we need to prove only that there exists an M-family **E** of V-sets such that $\bar{\mu}(X - X \cdot \sigma \mathbf{E}) < \epsilon$ and $\omega(\mathbf{E}, \psi) < \epsilon$.

For $n = 1, 2, \ldots$ we denote by $\bar{B}^*_n$ the set of $\bar{B}$-sequences $M_i(x)$ with $x \in X$, whose constituents belong to **V**, are included in G°_N, and satisfy

3.271 $$\rho(\Delta, 1, V) < n;$$

we denote the domain of $\bar{B}^*_n$ by X_n. Since the sequence $X_1, X_2, \ldots, X_n, \ldots$ is increasing with $\lim X_n = X$ (mod $\mathbf{N}^*$), we may and do choose ν so that

3.272 $$\mu(\bar{X}) - \mu(\bar{X}_\nu) < \epsilon;$$

we let $Z = X_\nu$.

We denote by $\mathbf{V}^1$ the spread of $\bar{B}^*_\nu$, by η a fixed number, $0 < \eta < 1$, by δ a positive number such that

3.273 $$0 < (\delta^{-1} - 1)\, \psi^\circ < \epsilon,$$

where $\psi^\circ = \sup \psi(M)$ for $\mathbf{M} \ni M \subset G_\nu$. Finally we fix an auxiliary sequence of positive numbers ϵ_i whose sum is less than ϵ.

Property (L) allows us to select an M-family $M_1, M_2, \ldots M_j, \ldots$ of V-sets such that

3.274 $$T = \bigcup_{j=1}^{\infty} M_j \supset Z (\text{mod } \mathbf{N}^*), \quad \mu(T - T \cdot \bar{Z}) < \epsilon^1$$

where $\epsilon^1 = \min \{\epsilon_1, \eta\, \bar{\mu}(Z)/2(\eta + \nu)\}$.

Since $\lim \mu(T_q) = \mu(T) < \infty$, where

$$T_q = \bigcup_{j=1}^{q} M_j,$$

we can choose Q so that

3.275 $$\mu(T - T_Q) < \epsilon^1.$$

Now, using 3.271, we can extract from the finite family $M_1, M_2, \ldots, M_Q$ a disjoint subfamily $M^1_1, M^1_2, \ldots, M^1_{q_1}$, such that for

$$T^1 = \bigcup_{k=1}^{q_1} M^1_k,$$

we have

3.276 $$\mu(T^1) > \mu(T_Q)/\nu.$$

This is the disentanglement step.

With each set M^1_k we associate an A-set $A^1_k \subset M^1_k$ in such a way that

3.277 $$\mu(A^1_k) > \eta\, \mu(M^1_k), \quad \psi(A^1_k) > \delta \psi(M^1_k).$$

We evaluate the Z-exhaustion of

$$S^1 = \bigcup_{k=1}^{q_1} A^1_k$$

by which we mean the value $\mu(S^1\cdot\bar{Z}) = \bar{\mu}(S^1\cdot Z)$. Since $S^1 \subset T$, $Z \subset T$ (mod $\mathbf{N}^*$), then, using 3.274, 3.275, 3.276, and 3.277, we derive

$$\begin{aligned}\mu(S^1\cdot\bar{Z}) &> \mu(\bar{Z}) + \mu(S^1) - \mu(T)\\ &> \bar{\mu}(Z) + \eta(\bar{\mu}(Z) - \epsilon^1)/\nu - (\bar{\mu}(Z) + \epsilon^1)\\ &= \eta\bar{\mu}(Z)/\nu - \epsilon^1(\eta + \nu)/\nu > \eta\bar{\mu}(Z)/2\nu.\end{aligned}$$

So far we have been able to select a finite disjoint subfamily $M^1_1, \ldots M^1_{q_1}$ of $\mathbf{V}^1$-sets (hence $\mathbf{V}$-sets), with a Z-overflow less than ϵ_1, and to contract each M^1_k into an $\mathbf{A}$-set A^1_k such that the new family has a Z-power of exhaustion greater than $\eta/2\nu$; that is, the *ratio of Z-exhaustion* is greater than $\zeta = \eta/2\nu$.

We repeat the process with $Y = Z - Z\cdot S^1$ and the family $\mathbf{V}^2$ consisting of those $\mathbf{V}^1$-constituents which do not intersect any A^1_k, $k = 1, 2, \ldots, q_1$, to produce two finite disjoint subfamilies $M^2_1, M^2_2, \ldots, M^2_{q_2}$ and $A^2_1, A^2_2, \ldots, A^2_{q_2}$ satisfying, for $l = 1, 2, \ldots, q_2$, $k = 1, 2, \ldots, q_1$, the relations

$$M^2_l \in \mathbf{V}^2,\ A^2_l \subset M^2_l,\ A^2_l \in \mathbf{A},\ A^1_k\cdot M^2_l = 0,\ \mu(S^2\cdot\bar{Y}) > \zeta\bar{\mu}(Y)$$

and

$$\mu(S^2 - S^2\cdot\bar{Y}) < \epsilon_2,$$

where

$$S^2 = \bigcup_{l=1}^{q_2} A^2_l.$$

The iteration of this exhaustion process yields two M-families, namely, $\mathbf{E}$ consisting of the sets $M^1_1, \ldots, M^1_{q_1}, M^2_1, \ldots, M^2_{q_2}, M^3_1, \ldots, M^3_{q_3}, \ldots$, and $\mathbf{C}$ consisting of the sets $A^1_1, \ldots, A^1_{q_1}, A^2_1, \ldots, A^2_{q_2}, A^3_1, \ldots, A^3_{q_3}, \ldots$ such that (i) the $\mathbf{E}$-constituents belong to $\mathbf{V}$, (ii) $S = \sigma\mathbf{E} \supset Z$ (mod $\mathbf{N}^*$), (iii) the overflow $\bar{\mu}(S - S\cdot\bar{Z}) < \epsilon$, (iv) to each $\mathbf{C}$-set A there corresponds an $\mathbf{E}$-constituent $V = \mathrm{D}(A)$ (the dilatation of A) satisfying $A \subset V$, $\psi(A) > \delta\psi(V)$. In view of these facts it is easily seen that the remainder of the proof involves merely a repetition of a portion of the proof of Theorem 3.24, hence $\mathbf{E}$ has met the two necessary requirements, and the theorem is proved.

Remarks. The proof of Theorem 3.26 is obtained from the preceding proof by discarding the contraction process.

Haupt's adaptation property is used only in the contraction process. If the weaker condition (UG) is substituted, then the assertion of the theorem remains true for Radon integrals.

In the proof of Theorem 3.24, Morse's function Δ is used to disentangle the infinite family of constituents to produce the desired countable family, hence the necessity of a choice condition based on the boundedness of Δ. In the foregoing proof we disentangle a finite family and repeat the process, producing the desired family by juxtaposition of sections.

In the proof of Theorem 3.27, we do not treat of Morse's halo $H(\Delta, \alpha, V_0)$ itself, but only with a finite family of halo constituents, which may be aptly

called a *partial halo*. For this reason we define $H'(\Delta, \alpha, V_0)$ as the union mod $\mathbf{N}$ (Definition 2.51) of the constituents V intersecting V_0, with $\Delta(V) < \alpha\Delta(V_0)$. In the formulation of Theorems 3.26 and 3.27 we can replace the halo dilatation $\rho(\Delta, \alpha, V_0)$ by

$$\rho'(\Delta, \alpha, V_0) = \mu(H'(\Delta, \alpha, V_0))/\mu(V_0)$$

clearly $\rho' < \rho$.

In defining H or H' we accept all constituents intersecting V_0 and satisfying $\Delta(V) < \alpha\Delta(V_0)$. The *incidence* requirement is $V \cdot V_0 \neq 0$. Correspondingly, disentanglement requires the determination of a *strictly disjointed* family of constituents. The incidence requirement may be altered to *essential intersection*, that is, $\mu(V \cdot V_0) > 0$, and in turn the disentanglement changed to require the production of a family of pairwise mod $\mathbf{N}$ disjointed constituents. This point of view can be adopted in Theorem 3.26 if we wish to achieve the strong Vitali property mod $\mathbf{N}$, and in Theorem 3.27, if we restrict the assertion to Radon integrals. The stronger we make the incidence requirement, the weaker the halo condition becomes. Busemann and Feller gave a necessary and sufficient halo condition for the validity of the density theorem, equivalent, by virtue of Theorem 2.12, to the Vitali μ-property, for Euclidean derivation bases of the $\mathfrak{D}$-type, the constituents being open sets, and the contraction being defined metrically. We have already encountered a halo of Busemann-Feller type in 2.4, namely $\sigma_{\alpha,r}(\mathbf{E})$.

3.3. Half-regular and regular branches of a derivation basis.

Definitions 3.31. To some of the sequences $M_\iota(x)$ of the basis $\bar{B}$ we correlate Moore-Smith sequences $M^*_\iota(x)$ of $\mathbf{M}$-sets of positive finite measure, with the same indices and the same convergence point, such that for each sequence

(R1) $$\liminf \mu(M^*_\iota)/\mu(M_\iota) > 0$$

(R2) $$\limsup \mu(M^*_\iota - M_\iota \cdot M^*_\iota)/\mu(M_\iota) = 0.$$

The set $\bar{B}^*$ of the sequences $M^*_\iota(x)$ is called a *half-regular branch* of $\bar{B}$, and $\bar{B}^* \cup \bar{B}$ is called a *half-regular extension* of $\bar{B}$. If (R2) is replaced by the strengthened requirement $M^*_\iota \subset M_\iota$, then $\bar{B}^*$ is called a *regular branch*, and $\bar{B}^* \cup \bar{B}$ is called a *regular extension* of $\bar{B}$. (**6**, 9.7).

De Possel has shown that the Vitali μ-property is preserved by half-regular extension. An example exists in which the sequences $M_\iota(x)$ are ordinary sequences of concentric closed squares in the plane (**10**, pp. 292–295). The corresponding set $M^*_\iota(x)$ consists of $M_\iota(x)$ augmented by small satellite squares, in such a manner that $\lim \mu(M^*_\iota)/\mu(M_\iota) = 1$, where μ denotes plane Lebesgue measure.

Example 3.23 deals with a regular branch of the basis of closed concentric intervals on the line. It shows that the strong Vitali property is not preserved under regular extension.

We shall now investigate the behavior of the Vitali properties under regular extension.

LEMMA 3.32. *We let ψ denote a Radon measure. We assume that there exists $\zeta > 0$ such that corresponding to any bounded set $X \subset E$ of positive outer measure, any $\bar{B}$-fine covering* **V** *of X, any μ-cover M of X, and any $\epsilon' > 0$, there exists an* M*-family* **E**$'$ *of* **V**-*constituents with union $S' = \sigma \mathbf{E}'$ for which*

$$\bar{\mu}(X \cdot S') > \zeta \bar{\mu}(X), \quad \psi(S' - S' \cdot M) < \epsilon', \quad \omega(\mathbf{E}', \psi) < \epsilon'.$$

Then $\bar{B}$ has the Vitali ψ-property.

Proof. We assume that $\epsilon > 0$ and attempt to find an M-family satisfying (V1), (V2), (V3) of Definitions 1.33. We introduce a sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers whose sum is less than $\epsilon/2$. Since X is bounded and **V** is a B-fine covering of X, we may and do assume that both X and all the **V**-sets lie in some $G^o{}_N$.

By hypothesis, there exists an M-family $\mathbf{E}_1$ of **V**-sets for which

$$\mu(X_1 \cdot S_1) > \zeta \bar{\mu}(X_1), \quad \psi(S_1 - S_1 \cdot M_1) < \epsilon_1, \quad \omega(\mathbf{E}, \psi) < \epsilon_1,$$

where $S_1 = \sigma \mathbf{E}_1$, $X_1 = X$, $M_1 = M$. We repeat this process on the sets $X_2 = X_1 - X_1 \cdot S_1$, $M_2 = M_1 - M_1 \cdot S_1$; by iteration in this way, we obtain a sequence of M-families $\mathbf{E}_1$, a nested sequence of sets $X_i \subset X$, with a nested sequence of μ-covers M_i, for $i = 1, 2, \ldots$, such that, for each such i, putting $S_i = \sigma \mathbf{E}_i$,

$$\mu(X_i \cdot S_i) > \zeta \bar{\mu}(X_i), \quad \psi(S_i - S_i \cdot M_i) < \epsilon_i, \quad \omega(\mathbf{E}_i, \psi) < \epsilon_i.$$

Finally, we let **E** denote the union of the families $\mathbf{E}_i$.

Since the rate of exhaustion at each step exceeds ζ, then **E** exhausts X; that is $S = \sigma \mathbf{E} \supset X$ (mod $\mathbf{N}^*$). We complete the proof by evaluating the ψ-overflow and ψ-overlap. We have

$$\psi(S - S \cdot M) < \sum_j \psi(S_j - S_j \cdot M_j) < \epsilon;$$

$$\begin{aligned}
\omega(\mathbf{E}, \psi) &= \sum_{V \in \mathbf{E}} \psi(V) - \psi(S) \\
&< \sum_j \Big(\sum_{V \in \mathbf{E}_j} \psi(V)\Big) - \psi(S \cdot M) = \sum_j \Big(\sum_{V \in \mathbf{E}_j} \psi(V) - \psi(S_j \cdot M_j)\Big) \\
&= \sum_j \Big(\sum_{V \in \mathbf{E}_j} (\psi(V) - \psi(S_j))\Big) + \sum_j \psi(S_j - M_j \cdot S_j) \\
&= \sum_j \omega(\mathbf{E}_j, \psi) + \sum_j \psi(S_j - M_j \cdot S_j) < \epsilon.
\end{aligned}$$

Remark. If Haupt's adaptation property holds, then the overflow requirement in the above may be omitted. The same is true if the weaker (UG) holds and ψ is a Radon μ-integral.

LEMMA 3.33. *If ψ is a Radon measure, $\tau = \psi + \mu$, $\bar{B}$ possesses the Vitali τ-property, and $\bar{B}^*$ is a regular branch of $\bar{B}$, then $\bar{B}^*$ possesses the Vitali ψ-property.*

Proof. We let X denote a subset of E of positive outer measure included in some G°_N, M a μ-cover of X, $\mathbf{V}^*$ a $\bar{B}^*$-fine covering of X, and ϵ a positive number.

For $n = 1, 2, \ldots$ we define $\bar{B}^*_n$ as the set of all $\bar{B}^*$-sequences S^* consisting of the sets $V^*_\iota(x)$ for which (i) $x \in X$, (ii) $V^*_\iota(x) \in \mathbf{V}^*$ for all ι, and (iii) there corresponds to each S^* a $\bar{B}$-sequence $\mathrm{D}(S^*) = S$, consisting of those sets $V_\iota(x)$ for which $V^*_\iota \subset V_\iota$ and $\mu(V^*_\iota)/\mu(V_\iota) > 1/n$ for all ι.

We denote by $\bar{B}_n$ the set of the $\bar{B}$-sequences associated by D to the $\bar{B}^*_n$-sequences, $\bar{B}_n = \mathrm{D}(\bar{B}^*_n)$, and by X_n the domain of $\bar{B}_n$, which is also the domain of $\bar{B}^*_n$. Since $\mathbf{V}^*$ is a $\bar{B}^*$-fine covering of X and $\bar{B}^*$ is a regular branch of $\bar{B}$, then X_n increases with n and $\lim \mu(\bar{X}_n) = \mu(\bar{X})$. As in the proof of Theorem 3.24 we may and do choose k so that

3.331 $$\mu(\bar{X}) - \mu(\bar{X}_k) < \epsilon.$$

We shall show by Lemma 3.32 that the subbasis $\bar{B}^*_k$ of $\bar{B}^*$ possesses the Vitali ψ-property. We consider a subset Y of X_k of positive outer measure, P a μ-cover of Y, $\bar{T}^*$ a subbasis of $\bar{B}^*_k$ with domain Y (mod $\mathbf{N}^*$), ϵ' a positive number. We define $\zeta = 1/2k$, and $\epsilon'' = \min(\epsilon', \bar{\mu}(Y)/4k)$. $\bar{T} = \mathrm{D}(\bar{T}^*)$ is a subbasis of $\bar{B}$ with the same domain as $\bar{T}^*$. Since $\bar{B}$ has the Vitali τ-property, there exists an M-family $\mathbf{E}$ of $\bar{T}$-sets such that

3.332 $$S = \sigma\mathbf{E} \supset Y(\mathrm{mod}\ \mathbf{N}^*), \quad \tau(S - S{\cdot}P) < \epsilon'', \quad \omega(\mathbf{E}, \tau) < \epsilon''.$$

With each $\mathbf{E}$-set V we associate a $\mathbf{V}^*$-set $V' = \mathrm{C}(V)$ (the *contraction* of V), with

3.333 $$\mathrm{C}(V) \subset V, \ \mu(V')/\mu(V) > 1/k.$$

We define the M-family $\mathbf{E}'$, demanded by Lemma 3.32, as $\mathrm{C}(\mathbf{E})$. Clearly

3.334 $$\omega(\mathbf{E}', \psi) \leqslant \omega(\mathbf{E}, \psi) \leqslant \omega(\mathbf{E}, \tau) < \epsilon'' \leqslant \epsilon';$$

the ψ-overlap condition is satisfied.

Putting $S' = \sigma\mathbf{E}'$,

3.335 $$\psi(S' - S'{\cdot}P) \leqslant \psi(S - S{\cdot}P) \leqslant \tau(S - S{\cdot}P) < \epsilon'' \leqslant \epsilon';$$

thus the ψ-overflow condition is satisfied.

We turn to the evaluation of the Y-exhaustion of ϵ', namely,

$$\mu(S'{\cdot}P) = \mu(S') - \mu(S' - S'{\cdot}P).$$

Since $\mu(S' - S'{\cdot}P) \leqslant \mu(S - S{\cdot}P) \leqslant \tau(S - S{\cdot}P) < \epsilon''$, we have $\mu(S'{\cdot}P) > \mu(S') - \epsilon''$. From the last relation in 3.332, $\omega(\mathbf{E}', \mu) \leqslant \omega(\mathbf{E}, \mu) < \epsilon''$, hence

$$\begin{aligned}\mu(S' \cdot P) &> \sum_{V' \in \mathbf{E}'} \mu(V') - 2\epsilon'' > (1/k) \sum_{V \in \mathbf{E}} \mu(V) - 2\epsilon'' \\ &\geqslant (1/k)\, \bar{\mu}(Y) - 2\epsilon'' \geqslant \bar{\mu}(Y)/2k = \zeta\bar{\mu}(Y).\end{aligned}$$

The basis satisfies the requirements of Lemma 3.32. Consequently, there exists an M-family $\mathbf{E}^*$ of $\bar{B}^*_n$-constituents, hence $\mathbf{V}^*$-constituents for which

$$S^* = \sigma\mathbf{E}^* \supset X_k (\text{mod } \mathbf{N}^*), \quad \psi(S^* - S^* \cdot M) < \epsilon, \quad \omega(\mathbf{E}^*, \psi) < \epsilon.$$

Due to the choice of k, from 3.331 we have obtained the Vitali ψ-property in the ϵ-version.

The following is an immediate consequence of Lemma 3.33.

THEOREM 3.34. *If $\bar{B}^*$ is a regular branch of $\bar{B}$, and $\bar{B}$ possesses the Vitali property for Radon measures (resp., integrals), then $\bar{B}^*$ possesses the Vitali property for Radon measures (resp., integrals).*

Examples 3.35. $\bar{B}$ is the cube basis in Euclidean space E_3, μ the Borel measure, $\mathbf{M}$ the family of Borel sets, $\bar{B}^*$ consists of all sequences of Borel sets of positive measure converging regularly to a point. According to Lemma 3.33, $\bar{B}^*$ possesses the Vitali property for Radon measures, hence it differentiates them.

The subbasis of $\bar{B}^*$ consisting of all sequences of closed sets of positive measure, converging regularly to a point, is the classical Lebesgue basis which enjoys the strong Vitali property. We may notice that any sequence of $\mathbf{M}$-sets of positive measure converging homothetically to a point belongs to $\bar{B}^*$.

If we denote by $\bar{B}^{**}$ the superbasis of $\bar{B}^*$ consisting of all sequences of Borel sets of positive measure converging to a point x (without regularity condition), it is easy to see that for a Radon integral $\psi(M) = \int_M f(M)\, d\mu$ we have $D^*\psi(x)$ = essential maximum of f at x, $D_*\psi(x)$ = essential minimum of f at x. It follows that the only Radon integrals differentiated by $\bar{B}^{**}$ are the integrals of functions summable at finite range and essentially continuous almost everywhere. This shows clearly the loss of differentiation power when discarding the regularity requirement for converging sequences. Finally, we notice that $\bar{B}^{**}$ is a blanket, different from both $\bar{B}^*$ and the Lebesgue basis.

3.4 Star blankets. To conclude, we give another example where the surrendering of the closeness assumption for the constituents of a basis means the replacement of the strong Vitali property by a Vitali property for Radon measures, with no loss of differentiation power.

DEFINITIONS 3.41. R denotes Euclidean n-dimensional space. The *hub* of a set $X \subset R$ is the set of these points $x \in X$ such that the segment joining x and x' lies in X whenever $x' \in X$; the *hub radius* of X at a point $x \in X$ is the supremum of those numbers ρ for which the solid sphere with x as center and ρ as radius is included in the hub of X. A *star blanket* (**15**, p. 432) according to Morse is a blanket F in R, whose constituents are closed sets, and such that for each x of its domain,

$$\limsup_{F(x) \ni V \to x} (\text{diam } V)/(\text{hub radius of } V \text{ at } x) < \infty.$$

We define *Borelian* star blankets by discarding in Morse's definition the

closeness requirement, and replacing it by the weaker demand that the constituents be Borel sets.

Remarks. Morse proved that a star blanket in his sense possesses the strong Vitali property *whenever μ is a Radon measure*. From his differentiation theorems he deduces the existence μ–almost everywhere of a finite derivative for every Radon measure ψ.

THEOREM 3.42. *Borelian star blankets possess the Vitali property for Radon measures, whenever the basic measure μ is itself a Radon measure.*

Sketch of the proof. Referring to **(15)**, all properties of star blankets exhibited in §5 up to the application of Morse's Theorem 3.4 in 5.11 are valid without using the fact of closeness of the constituents. The statement of this basic theorem, but for minor changes, is as follows: R is a metric space, μ a Radon measure, $0 < \zeta < \infty$, $X \subset R$, and $\mathbf{F}$ is a family of closed sets. Corresponding to each bounded open set G there exists a countable disjointed subfamily $\mathbf{K}$ of $\mathbf{F}$ for which $\sigma\mathbf{K} \subset G$, $\bar{\mu}(X \cdot G) < \zeta\mu(\sigma\mathbf{K})$. Then every disjointed subfamily of $\mathbf{F}$ can be extended to a countable subfamily of $\mathbf{F}$ covering X (mod $\mathbf{N}^*$).
If the $\mathbf{F}$-sets are required merely to be Borel sets, instead of closed, then we have to change the conclusion as follows: For any Radon measure ψ and any $\epsilon > 0$, every finite subfamily of $\mathbf{F}$ whose ψ-overlap is less than ϵ can be extended to a countable subfamily of $\mathbf{F}$ covering X (mod $\mathbf{N}^*$), and with ψ-overlap less than ϵ.

THEOREM 3.43. *Borelian star blankets differentiate every Radon measure, the basic measure μ being itself any Radon measure.*

Proof. This follows from Theorem 3.42.

Remark. This differentiation theorem, like Morse's theorems, implies that the set of points provided with μ-nullsequences is a μ-nullset.

§4. AN APPROACH TO A THEORY OF DIFFERENTIATION OF ABSTRACT INTERVAL FUNCTIONS

We take our setting as in 1.1. (G_e) and the reduced strong Vitali property are assumed to hold. λ denotes a finite numerical function defined on the spread $\mathbf{D}$.

4.1 Preliminary definitions.

DEFINITIONS 4.11. Any enumerable disjointed family $\mathbf{P}$ of $\bar{\mathcal{B}}$-constituents is called a *Vitali partition*. A partition $\mathbf{P}$ is called $\mathbf{V}$-*fine* if $\mathbf{V}$ is a family of $\bar{\mathcal{B}}$-constituents and the sets in $\mathbf{P}$ belong to $\mathbf{V}$. $\mathbf{P}$ is said to be *bounded* if the $\mathbf{P}$-constituents are included in some G°_N. If the sum

$$\sum_{V \in \mathbf{P}} \lambda(V)$$

represents a real number (including $+\infty$ and $-\infty$), we denote it by $\psi(\lambda, \mathbf{P})$ and we say that $\mathbf{P}$ is *λ-integrable.*

This last condition is always fulfilled when $\lambda \geqslant 0$. In general, however, λ may be of variable sign, and we shall henceforth assume that $\mathbf{P}$ is λ-integrable whenever $\mathbf{P}$ is a bounded Vitali partition.

DEFINITIONS 4.12. A *Vitali partition over* $X \subset R$ is defined as a Vitali partition covering X (mod $\mathbf{N}^*$).

A Vitali partition over X is thus a Vitali partition over any set Y having a μ-cover in common with X. From the reduced strong Vitali property it follows that for any bounded set $X \subset E$ and any $\tilde{B}$-fine covering $\mathbf{V}$ of X, there exists a $\mathbf{V}$-fine Vitali partition over X.

DEFINITION 4.13. A set X is called *λ-admissible* if any full $\tilde{B}$-fine covering of X includes a λ-integrable Vitali partition over X. According to our assumption above, the bounded subsets of E are λ-admissible.

DEFINITION 4.14. For any λ-admissible set $X \subset E$ we define the *upper Vitali integral* $\psi^\circ(X)$ or $\psi^\circ(\lambda, X)$ and the *lower Vitali integral* $\psi_0(X)$ or $\psi_0(\lambda, X)$ as lim sup $\psi(\lambda, \mathbf{P})$ and lim inf $\psi(\lambda, \mathbf{P})$, respectively, the limits being taken in the family of the λ-integrable partitions $\mathbf{P}$ over X with the full $\tilde{B}$-fine coverings of X serving as a scale of fineness. We have

$$\psi^\circ(X) = \psi^\circ(\tilde{X}\cdot E), \; \psi_0(X) = \psi_0(\tilde{X}\cdot E), \; \psi^\circ(N^*) = 0, \qquad N^* \in E\cdot\mathbf{N}^*.$$

λ is said to be *Vitali integrable* over X if $\psi^\circ(X) = \psi_0(X)$, *Vitali summable* if $\psi^\circ(X)$ and $\psi_0(X)$ are equal and finite. In either case the common value is denoted by $\psi(X)$.

Explicitly, λ is Vitali summable to $\psi(X)$ if $\psi(X)$ is finite, and corresponding to any $\epsilon > 0$, there exists a full $\tilde{B}$-fine covering $\mathbf{W}(\epsilon)$ of X such that for any λ-integrable Vitali partition $\mathbf{P}$ over X whose constituents belong to $\mathbf{W}(\epsilon)$, we have $|\psi(X) - \psi(\lambda, \mathbf{P})| < \epsilon$.

The Vitali integrals are discussed in **(18)**, wherein $E = R$, the strong Vitali property holds, and the $\tilde{B}$-constituents are assumed to be $\mathbf{G}$-sets. Thus the Vitali partitions over a set X form a directed system which is used as the scale of fineness. The definition for the Vitali integrals is subsumed by the more general one adopted here.

4.2 Differentiation theorems for Vitali summable functions.

LEMMA 4.21. *If $\tilde{B}$ has the strong Vitali property, X is a bounded subset of E, and α and β are any finite numbers, then:*

(a) *$\psi^\circ(X) \geqslant \alpha\bar{\mu}(X)$ whenever $D^*\lambda > \alpha$ almost everywhere on X;*

(b) *$\psi_0(X) \leqslant \beta\bar{\mu}(X)$ whenever $D_*\lambda < \beta$ almost everywhere on X.*

Proof. We first establish (a). If $\psi^\circ(X) = \infty$, there is nothing to prove. Accordingly, we assume that $\psi^\circ(X) < \infty$, $\epsilon > 0$, and $D^*\lambda > \alpha$ almost every-

where on X. In accordance with the definition of $\psi^\circ(X)$, there exists a full $\bar{B}$-fine covering $\mathbf{W}'$ of X such that for any bounded $\mathbf{W}'$-fine Vitali partition $\mathbf{P}$ over X, we have $\psi(\lambda, \mathbf{P}) < \psi^\circ(X) + \epsilon$. The family $\mathbf{V}'$ of the $\mathbf{W}'$ constituents satisfying the relation $\lambda(V) > \alpha\mu(V)$ is a $\bar{B}$-fine covering of X, hence, on account of the strong Vitali property and the boundedness of X, there exists a disjointed enumerable bounded subfamily $\mathbf{P}$ of $\mathbf{V}$ covering X (mod $\mathbf{N}^*$).

We have

$$\psi(\lambda, P) \geqslant \alpha \sum_{V \in \mathbf{P}} \mu(V) \geqslant \alpha\bar{\mu}(X).$$

Since $\mathbf{P}$ is $\mathbf{W}'$-fine, $\psi^\circ(X) > \psi(\lambda, \mathbf{P}) - \epsilon$. Combination of the two relations yields (a), since ϵ is arbitrary.

We turn to (b). If $\psi_0(X) = -\infty$, there is nothing to prove; we thus assume that $\psi_0(X) > -\infty$, $\epsilon > 0$, and $D_*\lambda < \beta$ almost everywhere on X. There exists a full $\bar{B}$-fine covering $\mathbf{W}''$ of X such that for any bounded $\mathbf{W}''$-fine Vitali partition $\mathbf{P}$, we have

$$\psi(\lambda, \mathbf{P}) > \psi_0(X) - \epsilon.$$

The family $\mathbf{V}''$ of the $\mathbf{W}''$-constituents satisfying $\lambda(V) < \beta\mu(V)$ is a $\bar{B}$-fine covering of X. Due to the strong Vitali property and the boundedness of X, $\mathbf{V}''$ includes a disjointed enumerable bounded subfamily $\mathbf{P}$ with

$$S = \sigma\mathbf{P} \supset X(\text{mod } \mathbf{N}^*), \ \mu(S - S\cdot\bar{X}) < \epsilon.$$

We have

$$\psi(\lambda, \mathbf{P}) \leqslant \beta \sum_{V \in \mathbf{P}} \mu(V) = \beta\mu(S) < \beta(\mu(\bar{X}) + \epsilon),$$

and since $\mathbf{P}$ is $\mathbf{W}''$ fine, $\psi(\lambda, \mathbf{P}) > \psi_0(X) - \epsilon$. Combining, we obtain $\psi_0(X) < \beta\bar{\mu}(X) + \epsilon(1 + \beta)$. From the arbitrary nature of ϵ, (b) follows.

THEOREM 4.22. *Assuming the strong Vitali property, if the function λ is Vitali summable over every bounded subset X of E, then the $\bar{B}$-derivative $D\lambda$ exists and is finite almost everywhere on E and $D\lambda$ is $\mu_E{}^*$-measurable. In particular, if on the bounded subsets X of E, $\psi(X)$ can be represented as*

$$\int_{\bar{X}\cdot E} f(x)\, d\mu_E$$

then $D\lambda = f$ (mod $\mathbf{N}^*$).

Proof. It follows readily from the preceding lemma that $N' = [D^*\lambda = \infty]$ and $N'' = [D_*\lambda = -\infty]$ are $\mathbf{N}^*$-sets.

We regard $D^*\lambda$ and $D_*\lambda$, restricted to the set $E^* = E - (N' \cup N'')$, as the functions f and g occurring in Lemma 1.23. Recalling the Remarks following Definitions 1.31, it clearly suffices to derive a contradiction from the assumed existence of two bounded μ^*-entangled sets A and B and two finite numbers α and β with $\beta < \alpha$, such that $D^*\lambda > \alpha$ on A and $D_*\lambda < \beta$ on B. However, since $\psi^\circ(A) = \psi^\circ(B) = \psi_0(B) = \psi_0(A)$, Lemma 4.21 yields the desired contradiction at once.

To prove the second part of the theorem, we let A and B denote two bounded subsets of E of positive measure, $A = B \pmod{\mathbf{N}^*}$, such that the convex closure of $f(A)$ and the convex closure of $D\lambda\ (B)$ are at a positive distance apart. This means that there exists two finite numbers α and β, with $\alpha > \beta$, and either

4.221 $$f(x) < \beta \text{ on } A,\quad D\lambda(x) > \alpha \text{ on } B$$

or

4.222 $$f(x) > \alpha \text{ on } A,\quad D\lambda(x) < \beta \text{ on } B.$$

Since the integrand f is $E\cdot\mathbf{M}$-measurable, then $f(x) < \beta$ almost everywhere on $\bar{A}\cdot E$ in case $f(x) < \beta$ on A, whence

$$\int_{\bar{A}.E} f(x)\, d\mu_E < \beta\mu_E(\bar{A}\,.\,E) = \beta\,\bar{\mu}(A);$$

while, according to the preceding lemma, $\psi(B) \geqslant \alpha\bar{\mu}(B)$ if $D\lambda(x) > \alpha$ on B. Since $\bar{\mu}(A) = \bar{\mu}(B) > 0$, and $\psi(A) = \psi(B)$, the inequalities in 4.221 are incompatible. Similarly, it follows that 4.222 cannot hold. Referring to Lemma 1.23 and the Remarks following Corollary 1.24, we obtain $D\lambda = f \pmod{\mathbf{N}^*}$.

4.3 An example of Vitali summable functions: The non-negative upper semi-additive functions.

DEFINITION 4.31. The non-negative function λ is called *upper semi-additive* on E (with respect to $\bar{B}$) if, corresponding to any $\bar{B}$-constituent V and any $\epsilon > 0$, there exists a full $\bar{B}$-fine covering $\mathbf{W}_\epsilon$ of $V\cdot E$ such that for any $\mathbf{W}_\epsilon$-fine Vitali partition $\mathbf{P}$, $\psi(\lambda, \mathbf{P}) < \lambda(V) + \epsilon$.

THEOREM 4.32. *If λ is a non-negative upper semi-additive function, then λ is Vitali integrable over the subset X of E and*

$$\psi(X) = \inf \sum_{V\in\mathbf{E}} \lambda(V)$$

for all M-*families* $\mathbf{E}$ *of $\bar{B}$-constituents*[5] *covering X* (mod $\mathbf{N}^*$). *In particular, if for any bounded Vitali partition* $\mathbf{P}$, *$\psi(\lambda, \mathbf{P})$ is finite, then $\psi(X)$ is finite on bounded X.*

Proof. We regard X as fixed and let

$$\gamma_0 = \inf \sum_{V\in\mathbf{E}} \lambda(V),$$

let ϵ denote an arbitrary positive number, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots$ a sequence of positive numbers whose sum is less than $\frac{1}{2}\epsilon$.

If $\gamma_0 = \infty$, then $\psi(X) = \infty$ and there is nothing to prove. We assume, then, that $\gamma_0 < \infty$. From the definition of γ_0, we are able to find an M-family $V_1, V_2, \ldots, V_n, \ldots$ of $\bar{B}$-constituents covering X (mod $\mathbf{N}^*$), such that

$$\sum_n \lambda(V_n) < \gamma_0 + \tfrac{1}{2}\epsilon.$$

[5]While the Vitali integral $\Psi(X)$ may be defined for vector valued functions, the right hand side of the equation just given presupposes a complete lattice structure.

For each V_n we determine a full $\tilde{\mathcal{B}}$-fine covering $\mathbf{W}_n$ in such a way that for any $\mathbf{W}_n$-fine Vitali partition $\mathbf{P}_n$ over $V_n \cdot E$,

$$\psi(\lambda, \mathbf{P}_n) < \lambda(V_n) + \epsilon_n,$$

and we define $\mathbf{W}^*$ as the union of the $\mathbf{W}_n$. $\mathbf{W}^*$ is a full $\tilde{\mathcal{B}}$-fine covering of X.

The theorem will be established if we prove that for any $\mathbf{W}^*$-fine Vitali partition $\mathbf{P}$ over X

$$\psi(\lambda, \mathbf{P}) < \gamma_0 + \epsilon,$$

since $\psi(\lambda, \mathbf{P}) \geqslant \gamma_0$. Accordingly, we let $\mathbf{P}$ be such a partition, and subdivide $\mathbf{P}$ into disjointed sub-partitions $\mathbf{P}_n$, $\mathbf{P}_n$ being $\mathbf{W}_n$-fine. This decomposition is certainly possible but not necessarily unique. Due to the upper semi-additivity of λ,

$$\psi(\lambda, \mathbf{P}_n) < \lambda(V_n) + \epsilon_n,$$

hence by addition

$$\sum_n \psi(\lambda, \mathbf{P}_n) = \psi(\lambda, \mathbf{P}) < \sum_n \lambda(V_n) + \tfrac{1}{2}\epsilon < \gamma_0 + \epsilon.$$

As for the second part of the theorem, if X is bounded, then X is covered (mod $\mathbf{N}^*$) by a bounded Vitali partition $\mathbf{P}$ for which $\psi(\lambda, \mathbf{P})$ is finite, therefore γ_0 is also finite.

Remark. Under the hypotheses of the theorem the function ψ defined on all subsets of E is a Carathéodory outer measure, meaning that it satisfies (C1) and (C2) of **(25**, p. 43**)**. Besides, ψ vanishes on the $\mathbf{N}^*$-subsets of E. With the Carathéodory restriction process of an outer measure to a measure in mind, we may expect the restriction of ψ to the $E \cdot \mathbf{M}$-sets to be a Radon μ_E-integral, which would enable us to apply Theorem 4.22. This proves true under the assumptions in **(18)**; more on this occurs in 4.4 below.

4.4 Morse's addivelous functions. Morse's definition of addivelous functions given when $\tilde{\mathcal{B}}$ is a Borelian blanket can be readily transposed to a general basis.

Definition 4.41. We say that the function λ defined on the family $\mathbf{D}^*$ of subsets of R is ***addivelous*** if:

(i) λ is non-negative;

(ii) $\mathbf{D}^*$ includes the spread $\mathbf{D}$ of $\tilde{\mathcal{B}}$;

(iii) Whenever the $\tilde{\mathcal{B}}$-constituents $V_1, V_2, \ldots$ are disjoint and included in the $\mathbf{D}^*$-set D^*, then $\Sigma\lambda(V_n) \leqslant \lambda(D^*)$;

(iv) if $V \in \mathbf{D}$ and $\epsilon > 0$, then there exists a $\mathbf{D}^*$-set $D = D(V)$ for which $\lambda(D) \leqslant \lambda(V) + \epsilon$, $I(D) \supset V \cdot E$ (mod $\mathbf{N}^*$).[6]

Defining $\mathbf{W}_\epsilon$ as the family of the $\tilde{\mathcal{B}}$-constituents included in D, we see that an addivelous function is upper semi-additive. The strong Vitali property secures the existence of a Vitali partition $V^\circ_1, V^\circ_2, \ldots,$ over E. We define (recall the final Remark after Definitions 1.31)

[6]For the definition of $I(D)$ refer to 1.1.

$$R^\circ_n = D(V_1) \cup D(V_2) \cup \ldots \cup D(V_n),$$

and regard a Vitali partition **P** as bounded if, for some N, the **P**-sets are included in $I\ (R^\circ_N)$. Thus for any bounded Vitali partition **P**, $\psi(\lambda, \mathbf{P})$ is finite.

Theorem 4.32 is applicable, and expresses the equivalence between Vitali integration and Morse's regularization, for the subsets of E.

In Morse's case, $\bar{B}$ is a Borelian blanket. Carathéodory's condition (C3) is clearly fulfilled. The restriction of ψ to the $E\cdot\mathbf{M}$-sets is a μ_E-integral, more precisely a Radon μ_E-integral with respect to the expanding reference sequence R°_n. Theorem 4.22 can be applied; $D\lambda$ is equal to a Radon-Nikodym μ_E-integrand of $\psi|E\cdot\mathbf{M}$. In an unpublished lecture delivered before the American Mathematical Society in 1948, Morse gave an interpretation, when $E = R$, of the indefinite integral of $D\lambda$ as $\psi|\mathbf{M}$, where $\psi(X)$ is defined as the infimum of numbers of the form

$$\sum_{V \in \mathbf{E}} \lambda(V),$$

where **E** is such a countable subfamily of **D** that **E** covers almost all of X.

References

1. H. Busemann and W. Feller, *Zur Differentiation der Lebegueschen Integrale*, Fundam. Math., *22* (1934), 226–256.
2. A. Denjoy, *Une extension du théorème de Vitali*, Amer. Jour. Math., *73* (1951), 314–356.
3. H. Hahn and A. Rosenthal, *Set Functions* (University of New Mexico, 1948).
4. P. Halmos, *Measure Theory* (New York, 1950).
5. O. Haupt, *Zum Beweise des Lebegueschen Ableitungssatzes*, Sitzungsberichten der Bayer. Akademie der Wissenschaften Mathematisch-naturwissenschaftliche Klasse No. 14 (1949), 171–174.
6. O. Haupt, G. Aumann, C. Pauc, *Differential- und Integral-rechnung*, 2nd ed., vol. III (Berlin).
7. O. Haupt and C. Y. Pauc, *Über die Ableitung absolut additiver Mengenfunktionen*, Archiv der Math., *1* (1948), 23–28.
8. ———, *Vitalische Systeme in Booleschen σ-Verbänden*, Sitzungsberichten der Bayer. Akademie der Wissenschaften, Mathematisch-naturwissenschaftliche Klasse, No. 14 (1950), 187–207.
9. ———, *Propriétés de mesurabilité de bases de dérivation*, Portugaliae Mathematica, *13* (1953), 37–54.
10. C. Hayes, *Differentiation with respect to φ-pseudo-strong blankets and related problems*, Proc. Amer. Math. Soc., *3* (1952), 283–296.
11. ———, *Differentiation of some classes of set functions*, Proc. Camb. Philos. Soc., *48* (1952), 374–382.
12. C. Hayes and A. P. Morse, *Some properties of annular blankets*, Proc. Amer. Math. Soc., *1* (1950), 107–126.
13. ———, *Convexical blankets*, Proc. Amer. Math. Soc. *1* (1950), 719–730.
14. A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc., *55* (1944), 205–235.
15. ———, *Perfect blankets*, Trans. Amer. Math. Soc., *61* (1947), 418–442.

16. A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc., *55* (1944), 236–305.
17. A. Papoulis, *On the strong differentiability of the indefinite integral*, Trans. Amer. Math. Soc., *69* (1950), 130–141.
18. C. Y. Pauc, *Compléments à la théorie de la dérivation de fonctions d'ensemble suivant de Possel et A. P. Morse*, C.R. Acad. Sci., Paris, *231* (1950), 1406–1408.
19. ———, *La dérivation dans les reseaux incomplets et les fonctions de Haar*, C.R. Acad. Sci., Paris, *232* (1951), 1387–1389.
20. ———, *Contributions à une théorie de la differentiation de fonctions d'intervalles sans hypothèse de Vitali*, C.R. Acad. Sci., Paris, *236* (1953), 1937–1939.
21. ———, *Ableitungsbasen, Prätopologie, und starker Vitalischer Satz*, J. reine angew. Math., *191* (1953), 69–91.
22. R. de Possel, *Dérivation abstraite des fonctions d'ensemble*, Journal de Math. pures et appliqués, *15* (1936), 391–409.
23. ———, *Sur la généralisation de la système dérivant*, C.R. Acad. Sci., Paris, *224* (1947), 1137–1139.
24. ———, *Sur les systèmes dérivants et l'extension du théorème de Lebesgue relatif à la dérivation d'une fonction de variation bornée*, C.R. Acad. Sci., Paris, *224* (1947), 1197–1198.
25. S. Saks. *Theory of the Integral* (Warsaw, 1937).
26. F. Wecken, *Abstrakte Integrale und festperiodische Funktionen*, Math. Zeit. *45* (1939), 377–404.
27. L. C. Young, *On area and length*, Fundam. Math. *35* (1948), 275–302.
28. B. Younovitch, *Sur les systèmes dérivants et l'extension du théorème de Lebesgue relatif à la dérivation d'une fonction de variation bornée*, Comptes rendus (Doklady) de l'Acad. des Sci. de l' U.R.S.S., *30* (1940), 112–114.
29. A. Zygmund, *On the differentiability of multiple integrals*. Fundam. Math., *23* (1934), 143–149.

University of California, Davis, Calif.

Université de Rennes, France

POINT-FINITE AND LOCALLY FINITE COVERINGS

ERNEST MICHAEL

1. Introduction. An interesting feature of recent topological developments is the increasingly important role played by locally finite coverings.[1] Point-finite coverings, on the other hand, even though conceptually simpler, have received very little attention. And deservedly so, since they are much less useful. Nevertheless, it sometimes happens (as it did to the author in **(5)**) that one is confronted by a covering which is known to be point-finite, but not necessarily locally finite. When does such a covering have a locally finite refinement? The purpose of this paper is to provide some answers to this question in the following two theorems (which the author happens to need in **(5)**), and to construct some counter-examples to certain related conjectures. It should be pointed out that, while Theorem 2 seems to be new, Theorem 1 is known **(6**, Theorem 3 and Lemma 3**)**, and is stated here only for completeness, and because it is needed in the proof of Theorem 2.

THEOREM 1 (Morita). *Every countable, point-finite covering of a normal space has a locally finite refinement.*

THEOREM 2. *Every point-finite covering of a collectionwise normal space has a locally finite refinement.*

Whether Theorem 1 remains true with "point-finite" omitted is one of the major unsolved problems in point-set topology, and it is equivalent to the problem of whether the cartesian product of a normal space and the closed unit interval is normal **(2**; **4)**. It is of course *not* possible to omit "point-finite" in Theorem 2, since a collectionwise normal space need not be paracompact. And finally, the following two counter-examples show that two other plausible directions for improving Theorem 2 are also barred:

Example 1. There exists a normal space, every point-finite covering of which has a locally finite refinement, but which is not collectionwise normal.

Example 2. There exists a normal space, not every point-finite covering of which has a locally finite refinement.

In §3, where these examples are constructed, it will be shown that they can even be slightly strengthened, and that, in particular, the spaces can be chosen to be perfectly normal.

We conclude this introduction with a quick review of our principal concepts. Let X be a *Hausdorff* space. In this paper, a *covering* of X is a collection of *open* subsets of X whose union is X. A collection $\mathscr{A}$ of subsets of X is *point-*

Received May 7, 1954.

1. All terms are defined at the end of this introduction.

finite if every $x \in X$ is an element of only finitely many $A \in \mathscr{A}$; it is *locally finite* if every $x \in X$ has a neighborhood which intersects only finitely many $A \in \mathscr{A}$. If $\mathscr{V}$ and $\mathscr{W}$ are coverings of X, then $\mathscr{W}$ is a *refinement* of $\mathscr{V}$ if every $W \in \mathscr{W}$ is a subset of some $V \in \mathscr{V}$. *Normal* spaces are, of course, familiar. According to Bing **(1)**, X is *collectionwise normal* if, whenever $\{A_\alpha\}$ is a collection of subsets of X which is *discrete* (i.e., locally finite, and with pairwise disjoint closures), there exists a disjoint collection $\{U_\alpha\}$ of open subsets of X such that $A_\alpha \subset U_\alpha$ for every α. Finally, X is *paracompact* if every covering of X has a locally finite refinement. The relations between these three types of spaces, as shown by Bing **(1)**, are that

$$\text{paracompact} \rightarrow \text{collectionwise normal} \rightarrow \text{normal},$$

and that neither arrow can be reversed.

2. Proof of Theorem 2. Let $\mathscr{U}$ be a point-finite covering of the collectionwise normal space X. We are going to construct a sequence $\{\mathscr{W}_i\}$ $(i = 0, 1, \ldots)$ of collections of open subsets of X such that, denoting $\bigcup\{W | W \in \mathscr{W}_i\}$ by W_i, the following conditions are satisfied for all i:

(a) Every $W \in \mathscr{W}_i$ is a subset of some $U \in \mathscr{U}$.

(b) $\mathscr{W}_i$ is locally finite (in fact, discrete).

(c) If $x \in X$ is an element of at most i elements of $\mathscr{U}$, then

$$x \in \bigcup_{k=0}^{i} W_k.$$

(d) Every $x \in W_i$ is an element of at least i elements of $\mathscr{U}$.

Suppose, for a moment, that $\{\mathscr{W}_i\}$ has been constructed, and notice how the theorem follows. In fact, remembering that $\mathscr{U}$ is point-finite, we see that $\{W_i\}$ is a covering of X (by (c)) which is point-finite (by (d)). It then follows from Theorem 1 that $\{W_i\}$ has a locally finite refinement $\{V_i\}$, with $V_i \subset W_i$ for every i, and therefore $\bigcup_{i=0}^{\infty}\{V_i \cap W | W \in \mathscr{W}_i\}$ is a locally finite refinement of $\mathscr{U}$ (by (a) and (b)).

It remains to construct the sequence $\{\mathscr{W}_i\}$. Let $\mathscr{W}_0 = \{\phi\}$ (i.e., the only element of $\mathscr{W}_0$ is the null set); then conditions (a)-(d) are clearly satisfied for $i = 0$. Suppose, therefore, that $\mathscr{W}_0, \ldots, \mathscr{W}_n$ have been constructed to satisfy (a)-(d) for all $i \leq n$, and let us construct $\mathscr{W}_{n+1}$.

Let $\mathfrak{R}$ be the family of all $\mathscr{R} \subset \mathscr{U}$ such that $\mathscr{R}$ has exactly $n + 1$ elements. For every $\mathscr{R} \in \mathfrak{R}$, let

$$A(\mathscr{R}) = \left(X - \bigcup_{k=0}^{n} W_k\right) \cap \left(X - \bigcup\{U \in \mathscr{U} | U \notin \mathscr{R}\}\right)$$

Clearly every $A(\mathscr{R})$ is closed. Let us show that $\{A(\mathscr{R}) | (\mathscr{R}) \in \mathfrak{R}\}$ is discrete, by showing that every $x \in X$ has a neighborhood which intersects at most one $A(\mathscr{R})$. We consider three cases: if x is in $> n + 1$ elements of $\mathscr{U}$, then the

intersection of any $n+2$ of these does not intersect any $A(\mathscr{R})$; if x is in $< n+1$ elements of $\mathscr{U}$, then (by (c))

$$x \in \bigcup_{k=0}^{n} W_k,$$

which does not intersect any $A(\mathscr{R})$; and if, finally, x is in exactly $n+1$ elements of $\mathscr{U}$, say in $U_1, \ldots, U_{n+1}$, then

$$\bigcap_{k=1}^{n+1} U_k$$

is a neighborhood of x which does not intersect $A(\mathscr{S})$ for $\mathscr{S} \neq \{U_1, \ldots, U_{n+1}\}$ (since then at least one U_k is not an element of $\mathscr{S}$, and this U_k cannot intersect $A(\mathscr{S})$).

Since $\{A(\mathscr{R})|\mathscr{R} \in \mathfrak{R}\}$ is thus a discrete collection of closed subsets of the collectionwise normal space X, there exists a disjoint collection $\{V(\mathscr{R})|\mathscr{R} \in \mathfrak{R}\}$ of open subsets of X such that $A(\mathscr{R}) \subset V(\mathscr{R})$ for every $\mathscr{R} \in \mathfrak{R}$; by a result of Dowker (**3**, p. 308), we can even pick $\{V(\mathscr{R})|\mathscr{R} \in \mathfrak{R}\}$ to be discrete. Now notice that $A(\mathscr{R}) \subset U$ for every $U \in \mathscr{R}$, since otherwise some $x \in A(\mathscr{R})$ would be an element of $<n$ elements of $\mathscr{U}$, which is impossible by (c) and the definition of $A(\mathscr{R})$. Hence, if we let

$$P(\mathscr{R}) = V(\mathscr{R}) \cap \bigcap \{U|U \in \mathscr{R}\},$$

then $A(\mathscr{R}) \subset P(\mathscr{R})$ for every $\mathscr{R} \in \mathfrak{R}$. We now define $\mathscr{W}_{n+1} = \{P(\mathscr{R})|\mathscr{R} \in \mathfrak{R}\}$. Let us check that conditions (a)-(d) are satisfied for $i = n+1$. That (a), (b), and (d) are satisfied follows directly from the definition of $\mathscr{W}_{n+1}$. To check (c), let $x \in X$ be an element of $<n+1$ elements of $\mathscr{U}$; then clearly there exists an $\mathscr{R} \in \mathfrak{R}$ such that $x \in (X - \bigcup\{U \in \mathscr{U}|U \notin \mathscr{R}\})$. But then either

$$x \in \left(X - \bigcup\{U \in \mathscr{U}|U \notin \mathscr{R}\}\right) \cap \left(X - \bigcup_{k=0}^{n} W_k\right) = A(\mathscr{R}) \subset P(\mathscr{R}) \subset W_{n+1},$$

or
$$x \in \bigcup_{k=0}^{n} W_k;$$

thus in either case

$$x \in \bigcup_{k=0}^{n+1} W_k.$$

This completes the proof.

3. The Counter-examples In this section we shall describe the spaces of Examples 1 and 2 in the introduction, and show that they have the required properties:

Example 1. As a space with the required properties, we submit the normal, but not collectionwise normal, space F of Bing (**1**, Example G). We refer the reader to Bing's paper for the definition of F, and for the related notation.

We shall use Bing's notation, adding one additional piece of notation of our own: If $p \in P$, and if r is a finite subcollection of Q, then

$$\langle p, r\rangle = \{f \in F \mid f(q) = f_p(q) \quad \text{for all } q \in r\}.$$

We must show that every point-finite covering of F has a locally finite refinement. So let $\mathscr{U}$ be a point-finite covering of F. Let $\mathscr{V} = \{U \in \mathscr{U} \mid U \cap F_p \neq \phi\}$. There are now two possibilities:

(a) $\mathscr{V}$ is countable. Let $V = \bigcup\{V \mid V \in \mathscr{V}\}$. Then V is an open and closed subset of F, and is therefore normal. Hence $\mathscr{V}$ is a countable, point-finite covering of the normal space V, and hence (by Theorem 1) has a locally finite refinement $\mathscr{R}$. If we now let $\mathscr{S} = \mathscr{R} \cup \{\{f\} \mid f \in (F - V)\}$, then $\mathscr{S}$ is a locally finite refinement of $\mathscr{U}$.

(b) $\mathscr{V}$ is uncountable. We shall show that this is impossible. For suppose it is true. Then, it is easy to check, there exists an uncountable subset M of P, and for each $p \in M$ a finite subcollection r_p of Q, such that the family of all $\langle p, A_p\rangle$, with $p \in M$, is point-finite. Bing's proof that F is not collectionwise normal actually proves that such a family cannot be *disjoint:* the proof that it cannot even be *point-finite* is very similar, and we therefore only indicate the necessary modification in Bing's proof. Bing begins by assuming that the collection of all $\langle p, r_p\rangle$ is disjoint, and obtains his contradiction by finally showing that it isn't even point-finite. The only place where Bing actually uses the disjointness of $\{\langle p, r_p\rangle\}_{p \in M}$ is, essentially, to show the existence of an uncountable $W_1' \subset W$ (where W is an uncountable subset of M), and a $q_1 \in Q$, such that $q_1 \in r_p$ for every $p \in W_1'$. To show the existence of q_1 and W_1' even under the weaker assumption that $\{\langle p, r_p\rangle\}_{p \in M}$ is point-finite, we proceed as follows: Let T be a maximal subset of W having the property that $r_p \cap r_{p'} = \phi$ whenever $p \in T, p' \in T, p \neq p'$; the existence of such a set follows from Zorn's lemma. It is easy to see that

$$\bigcap_{p \in T}\langle p, r_p\rangle \neq \phi,$$

and hence T must be finite. Letting $r = \bigcup_{p \in T} r_p$, we see that r is a finite subcollection of Q. Now for every $q \in r$, let $E_q = \{p \in W \mid q \in r_p\}$; it follows from the maximality of T that $\bigcup_{q \in r} E_q = W$. Hence E_q must be uncountable for at least one $q \in r$, say for q_1. If we now let $W_1' = E_{q_1}$, then W_1' and q_1 have the required properties.

To obtain a space, satisfying our requirements, which is also perfectly normal (i.e., every closed subset is a G_δ), we need only replace the above space F of Example G of **(1)** by the space F of Example H of **(1)**. The proof goes just as before.

Example 2. We shall construct a normal, non-collectionwise normal space G, every covering of which has a point-finite refinement. (This last property is sometimes called pointwise paracompactness.) This space certainly has all required properties, since if every point-finite covering of G had a locally

finite refinement, it would follow that G is paracompact, and hence collectionwise normal, which it is not.

To obtain G, we begin with the space F of Bing (**1**, Example G) which was used in Example 1, and then let G be the subspace of F defined by

$$G = F_p \cup \{f \in F \mid f(q) = 0 \text{ except for finitely many } q \in Q\}.$$

Since G is a closed subset of F, G is normal. Bing's proof that F is not collectionwise normal goes through *verbatim* to show that G is not collectionwise normal. All that remains to show is that every covering of G has a point-finite refinement.

Let $\mathcal{U}$ be a covering of G. For each $p \in P$, pick a $U_p \in \mathcal{U}$ such that $f_p \in U_p$, and let $V_p = \{f \in G \mid f(p) = 1\}$. It follows from the definition of G that $\{V_p\}_{p \in P}$ is point-finite. If we now let

$$\mathcal{W} = \left(\{W_p \cap U_p\}_{p \in P}\right) \cup \left(\{\{f\}\}_{f \in G - F_p}\right),$$

then $\mathcal{W}$ is clearly a point-finite refinement of $\mathcal{U}$. This completes the proof that G has all the required properties.

Just as in Example 1, we can obtain a space, satisfying all our requirements, which is also perfectly normal. In fact, all we need to do is to start with the space F of Example H of (**1**), rather than with the space F of Example G of (**1**). We then let

$$G = F_p \cup \{f \in F \mid f(q) \text{ is even except for finitely many } q \in Q\}.$$

The proof that G does the trick proceeds just as before.

References

1. R. H. Bing, *Metrization of topological spaces*, Can. J. Math., *3* (1951), 175–186.
2. C. H. Dowker, *On countably paracompact spaces*, Can. J. Math., *3* (1951), 219–224.
3. ———, *On a theorem of Hanner*, Ark. Mat., *2* (1952), 307–313.
4. M. Katětov, *On real-valued functions in topological spaces*, Fund. Math., *38* (1951), 85–91.
5. E. Michael, *Continuous selections* I, to appear in Annals of Mathematics.
6. K. Morita, *Star-finite coverings and the star-finite property*, Math. Japonicae, *1* (1948), 60–68.

University of Washington, Seattle

ON DYNAMICAL SYSTEMS WITH ONE DEGREE OF FREEDOM

C. R. PUTNAM

1. Introduction. Consider the (vector, n-component) system of differential equations

$$x' = f(x) \quad (' = d/dt), \tag{1}$$

where $f(x)$ is of class C^1. Let Ω denote a set of points, x, consisting of unrestricted solution paths $x(t)$, so that the $x(t)$ exist and lie in Ω for $-\infty < t < \infty$. Let $t = t_0$ be arbitrary but fixed; then the solution $x = x(t)$ will be called stable (with respect to Ω) if for every $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ such that $|x(t) - y(t)| < \epsilon$ whenever $y(t)$ is in Ω and $|x(t_0) - y(t_0)| < \delta$. For a discussion of this type of stability (called A-stability in **(7)**), see Liapounoff **(4**, pp. 210–211; **8**, pp. 98–99**)**, wherein are given references to Minding and Dirichlet.

It was shown by Hartman and Wintner **(3)** that a solution $x(t)$ of (1) which is dense on a compact set Ω is almost periodic (in the sense of Bohr) if and only if it is stable in a certain sense. The type of stability considered there (called B-stability in **(7)**), however, is more restrictive than the A-stability, and, in the sequel, the term "stability" will refer only to that (A-stability) defined at the beginning of this section. It was shown in **(7)** that if (1) is of the incompressible type, so that

$$\operatorname{div} f \equiv \Sigma \partial f_k/\partial x_k = 0, \tag{2}$$

and if the space Ω is suitably restricted, then all stable solutions of (1) do have certain properties possessed by almost periodic solutions. Whether all such solutions, for n arbitrary, are actually almost periodic will remain undecided. (If (2) is not assumed, then stability surely does not imply almost periodicity even if the dimension number n of (1) is unity; see **(7)**.)

The present paper will be devoted to a consideration of (1), subject to the (measure-preserving) condition (2), in the special case when $n = 2$. It is known that the system (1) is then equivalent to a conservative Hamiltonian system **(8**, p. 88**)**, and hence, in view of the existence of the energy integral, is completely integrable. In what follows then, only Hamiltonian systems of one degree of freedom, that is, systems of the type

$$p' = -\partial H/\partial q,\ q' = \partial H/\partial p \quad (H = H(p, q),\ ' = d/dt), \tag{3}$$

will be considered. The identification with (1) is $x_1 = p, x_2 = q, f_1 = -\partial H/\partial q$,

---

Received July 30, 1954. This work was supported in part by the National Science Foundation research grant NSF-G481.

$f_2 = \partial H/\partial p$; the assumption that $f(x)$ of (1) be of class C^1 is now that the Hamiltonian $H(p, q)$ be of class C^2. The following theorem will be proved:

I. *Let $H(p, q)$ be of class C^2 on the p, q space, and let Ω denote an unrestricted invariant set of finite positive measure. Then through every point $x = (p, q)$ of Ω, except possibly for those belonging to a set Z of measure* 0, *there exists either an equilibrium solution* ($x \equiv$ const.) *or a periodic solution of* (3).

It will turn out that the set Z is the set excluded from the assertion of the Poincaré recurrence theorem on Ω (see §3 below). It was shown in **(7)**, however, that, even for general systems (1) satisfying (2) with dimension number n arbitrary, if the set Ω satisfies

$$\text{meas}\,\Omega\Sigma > 0, \tag{4}$$

where x is an arbitrary point of Ω and Σ is any open sphere (disk, in the present case) with center at x, then no point of a stable path can belong to Z. As a consequence, there follows the theorem

II. *Under the same assumptions as in* (I), *along with the additional condition* (4), *every stable solution path $x(t) = (p(t), q(t))$ is periodic (possibly constant) on $-\infty < t < \infty$.*

Needless to say, the condition (4) is fulfilled if, for instance, the set Ω is open or is the closure of an open set.

It should be noted that the existence of a closed (Jordan) curve in the p, q space of the form $H =$ const. does not necessarily imply that this is the path of a periodic solution of (3). One need only consider the physical example of a simple pendulum oscillating with an energy just sufficient to raise the pendulum (asymptotically) to its greatest possible height, corresponding to a position of unstable equilibrium.

For a general discussion of systems (1) when $n = 2$, see the series of papers of Poincaré **(5)**, especially the one of 1885. It should be noted that the notion of stability considered there **(5**, pp. 167–172**)** is not that of the present paper, but what is sometimes termed stability in the sense of Poisson. In this connection, compare the recurrence theorem of Poincaré **(6**, pp. 67 ff**)** cited at the beginning of §3 below. For further references to the case $n = 2$, see Birkhoff **(1)**, in particular pp. 123–124, and Brown **(2)**.

Another corollary of I is

III. *Let $H = H(p, q)$ be of class C^2 and suppose that the point (p_0, q_0) is an isolated equilibrium point of the system* (3). *If (p_0, q_0) is a stable point (that is, if the solution $p \equiv p_0$, $q \equiv q_0$ is stable), then it is either a (local) maximum or a minimum point of the funtion $H(p, q)$.*

The question as to whether there is a theorem corresponding to III for the case of a conservative dynamical system with n degrees of freedom was pointed out by Wintner **(8**, p. 101**)** and will remain undecided. It is known

that the conclusion of III can become false if the restriction that (p_0, q_0) be an isolated equilibrium point is dropped (**8**, pp. 100–101).

The proof of III as a consequence of Theorem I will be given in §2; the proof of I will be given in §3.

2. Proof of III. Grant Theorem I. Since $x_0 = (p_0, q_0)$ is a stable equilibrium point, there exists a sequence of invariant (and, if desired, open) sets containing, and closing down upon, the point x_0 (Poincaré-Birkhoff criterion). Choose one of these sets and call it Ω; it is clear that the assumptions of I are now fulfilled.

Since x_0 is an isolated equilibrium point, it follows from I that through almost all points sufficiently close to x_0 there exist (non-constant) periodic solution of (3), corresponding to closed Jordan curves in the p, q space. Consider a sequence of such curves $C_1, C_2, \ldots$, which, in view of the stability assumption on x_0, tend to the point x_0. Since the energy integrals $H =$ const. of (2) are the equations of the solutions in the p, q plane, each of the curves C_n is a level curve of H. Hence the function H attains either a (local) maximum or a minimum value at at least one point, say x_n, *inside* each (Jordan) curve C_n. Clearly the x_n are equilibrium points and satisfy $x_n \to x_0$ as $n \to \infty$. Since x_0 is an isolated equilibrium point, then $x_n = x_0$ for n sufficiently large (and hence for some one value of n). Consequently, x_0 is either a maximum or a minimum point of H and the proof of III is complete.

3. Proof of I. The assumptions on Ω are those guaranteeing the validity of the Poincaré recurrence theorem (cf., in this connection, **6**, pp. 67 ff.; **8**, p. 91; **7**). Let Z denote the set of measure zero excluded from the assertion of this theorem, so that, if x_0 is not in Z, the solution path $x(t)$ through x_0 has the following property: if t^* is arbitrary, there exists a sequence of dates t_n, where $n = \pm 1, \pm 2, \ldots$, such that $t_n \to \infty$ or $-\infty$ according as $n \to \infty$ or $-\infty$ and $x(t_n) \to x(t^*)$ as $|n| \to \infty$. It will be shown that if $x_0 = (p_0, q_0)$ is in $\Omega - Z$, then the (vector) function $x(t)$, where $x_0 = x(t_0)$, is periodic or constant (for $-\infty < t < \infty$).

Suppose, if possible, that $x(t) \not\equiv$ const. and not periodic. Since $x_0 = x(t_0)$ is in $\Omega - Z$, there exist values t_n and points $x_n = x(t_n)$ such that $x_n \neq x_0$ and $x_n \to x_0$ as $n \to \infty$. Since $x(t) \not\equiv$ const., so that x_0 is not an equilibrium point, the general existence theorem along with the attending continuity properties for solutions of ordinary differential systems (1), guarantees the existence of points X_n on the curve $x(t)$ which lie on the normal line to this curve at the point x_0, and satisfy $X_n(\neq x_0) \to x_0$. Since the solution path curve $x(t)$ constitutes a branch of the locus $H(p, q) = c$, for some constant c, it is clear that the directional derivatives of H, taken along the tangent and the normal to the path $x(t)$ at the point x_0, are zero. Consequently, the vector grad H is zero at this point; and so x_0 is an equilibrium point, in contradiction with the supposition at the beginning of this paragraph. This completes the proof of I.

REFERENCES

1. G. D. Birkhoff, *Dynamical systems* (New York, Amer. Math. Soc., 1927).
2. A. B. Brown, *Relations between the critical points and curves of a real analytic function of two independent variables*, Ann. Math., *31* (1930), 449–456.
3. P. Hartman and A. Wintner, *Integrability in the large and dynamical stability*, Amer. J. Math., *65* (1943), 273–278.
4. A. Liapounoff, *Problème générale de la stabilité du mouvement*, Ann. Math. Studies, no. 17 (Princeton, 1947).
5. H. Poincaré, *Sur les courbes définies par une équation différentielle*, J. de Math., ser. 3, *7* (1881), *8* (1883), ser. 4, *1* (1885), *2* (1886).
6. ———, *Sur le problème des trois corps et les équations de la dynamique*, Acta Math., *13* (1890).
7. C. R. Putnam, *Stability and almost periodicity in dynamical systems*, Proc. Amer. Math. Soc., *5* (1954), 352–356.
8. A. Wintner, *The analytical foundations of celestial mechanics* (Princeton, 1941).

Purdue University

THE PROPAGATION OF A PLANE SHOCK INTO A QUIET ATMOSPHERE II

M. H. MARTIN

1. Introduction. In a previous paper **(3)** the study of one-dimensional, unsteady flows, isentropic or anisentropic, was reduced to the integration of a Monge-Ampère partial differential equation

$$\xi_{\psi\psi}\xi_{pp} - \xi^2_{\psi p} = \tau_p. \tag{1}$$

For a polytropic gas, the specific volume

$$\tau = e^{(S-S_0)/c_p} \cdot p^{-n}, \qquad n = 1/\gamma, \tag{2}$$

takes the form

$$\tau = \delta(\psi)p^{-n},$$

once the *entropy distribution function* $S = S(\psi)$ is specified. A solution $\xi = \xi(\psi, p)$ having been determined by one means or another, the actual flow is presented by the mapping

$$x = \int \{(\xi_{\psi\psi}\xi_{\psi p} + \tau)\, d\psi + \xi_\psi\xi_{pp} dp\}, \qquad t = \xi_p, \ (u = \xi_\psi), \tag{3}$$

of the (ψ, p)-plane upon the (x, t)-plane. This mapping carries the rectilinear network $\psi = \text{const.}$, $p = \text{const.}$, in the (ψ, p)-plane into the curvilinear network of trajectories and isobars in the (x, t)-plane.

A progressive condensation shock, carrying in back of it the values u, τ, p of the velocity, specific volume and pressure and moving into a quiet atmosphere where these quantities have fixed values u_0, τ_0, p_0 is governed by the shock conditions:

$$u = u_0 + \sqrt{(p - p_0)(\tau_0 - \tau)}, \tag{4.1}$$

$$\frac{dx}{dt} = u_0 + \tau_0\sqrt{\frac{p - p_0}{\tau_0 - \tau}}, \tag{4.2}$$

$$\tau = \tau_0 \frac{(\gamma - 1)p + (\gamma + 1)p_0}{(\gamma + 1)p + (\gamma - 1)p_0}, \tag{4.3}$$

where (4.2) gives the shock velocity.

Once the entropy distribution function $S(\psi)$ is selected, the determination of the motion of the shock into the quiet atmosphere and of the flow immediately behind it, sets **(3)** a Problem of Cauchy for the controlling partial differential equation (1).

Received August 28, 1954. This work was carried out under Contract DA-36-034-ORD-1486 with the Office of Ordnance Research, U.S. Army.

If, however, one is willing to forgo knowledge of the flow immediately in back of the shock, the motion of the shock into the quiet atmosphere can be determined without solving the Problem of Cauchy. In this paper we investigate the influence of the choice of the entropy distribution function $S(\psi)$ upon the propagation of the shock into the quiet atmosphere. Among other things we find that the time required (2, p. 213) for complete decay of the shock may be finite or infinite, depending upon the selection of the entropy distribution function.

2. The shock curve. If we substitute from (2) for τ in (4.3) and prescribe an entropy distribution function $S(\psi)$, the resulting equation

$$S(\psi) - S_0 = c_v \log\left(\tau_0 \frac{(\gamma - 1)p + (\gamma + 1)p_0}{(\gamma + 1)p + (\gamma - 1)p_0} p^{\gamma}\right) \tag{5}$$

defines a curve $\psi = \psi(p)$ in the (ψ, p)-plane. This curve transforms by (3) into the *shock curve* in the (x, t)-plane and we propose to deduce its parametric equations

$$x = x(p), \qquad t = t(p), \tag{6}$$

for a given entropy distribution function $S(\psi)$. Here p denotes the pressure immediately in back of the shock.

From (3) we see that a curve $\psi = \psi(p)$ in the (ψ, p)-plane is carried into a curve $x = x(t)$ in the (x, t)-plane along which

$$\frac{dx}{dt} = u + \tau \frac{d\psi}{dp} \bigg/ \frac{dt}{dp}. \tag{7}$$

As a matter of fact this result is an immediate consequence of the relation

$$d\psi = \rho\, dx - \rho u\, dt, \tag{8}$$

i.e., of the principle of conservation of mass.

We take (5) for the curve $\psi = \psi(p)$, substitute in (7) for u and dx/dt from (4.1) and (4.2), to obtain

$$\frac{dt}{dp} = \frac{d\psi}{dp} \sqrt{\left(\frac{\tau_0 - \tau}{p - p_0}\right)}. \tag{9}$$

When τ is eliminated from this by (4.3) we find

$$t = \int \frac{d\psi}{dp} \sqrt{\left(\frac{2\tau_0}{(\gamma + 1)p + (\gamma - 1)p_0}\right)}\, dp, \tag{10}$$

to reduce the determination of the function $t(p)$ in (6) to a quadrature, once the function $\psi = \psi(p)$ is fixed by the selection of $S(\psi)$ in (5).

To determine the function $x(p)$ in (6), we write (4.2) in the form

$$\frac{dx}{dp} = \left[u_0 + \tau_0 \sqrt{\frac{p - p_0}{\tau_0 - \tau}}\right] \frac{dt}{dp},$$

and substitute in here for dt/dp from (9). This yields

$$x = \tau_0\psi + u_0 t, \tag{11}$$

where ψ, t are the functions of p defined in (5) and (10). This result may be checked by applying the principle of conservation of mass across the shock in the form

$$d\psi = \rho dx - \rho u dt = \rho_0 dx - \rho_0 u_0 dt.$$

We sum up our results on the shock curve in the theorem below.

THEOREM. *For a shock moving into a quiet atmosphere in which the velocity u, specific volume τ, and pressure p have fixed values u_0, τ_0, p_0, the shock curve in the (x, t)-plane is described parametrically in terms of the pressure p immediately in back of the shock by*

$$x = \tau_0\psi + u_0 t, \quad t = \int \frac{d\psi}{dp}\sqrt{\left(\frac{2\tau_0}{(\gamma+1)p + (\gamma-1)p_0}\right)}dp, \tag{12}$$

in which the function $\psi(p)$ is defined implicitly by

$$S(\psi) = c_p \log\left(\tau_0 \frac{(\gamma-1)p + (\gamma+1)p_0}{(\gamma+1)p + (\gamma-1)p_0} p^n\right) + S_0, \qquad S_0 = \text{const.},$$

upon prescription of the entropy distribution function $S(\psi)$.

3. Shock decay. The problem of shock decay is of some interest (**1**; **2**) and we shall make some remarks on how decay is affected by the choice of the entropy distribution function.

First of all, for a shock to decay completely it is necessary and sufficient that the pressure p in back of the shock equal the pressure p_0 in front of the shock.

Starting with a pressure $p_1 > p_0$ in back of the shock, it is clear from the above theorem that complete decay will require a finite or an infinite time, according as the integral

$$t(p_0) - t(p_1) = \int_{p_1}^{p_0} \frac{d\psi}{dp}\sqrt{\left(\frac{2\tau_0}{(\gamma+1)p + (\gamma-1)p_0}\right)}\, dp \tag{13}$$

converges or diverges.

To fix the ideas, let us take $u_0 = 0$, so that the gas in front of the shock is at rest, and assume that the entropy distribution function $S(\psi)$ is chosen so that

$$-\infty < \frac{d\psi}{dp} < 0 \quad \text{for} \quad p_1 < p < p_0. \tag{14}$$

From the parametric equations (12) of the shock curve it is clear that x and t both increase monotonically for decreasing p as shown in Fig. 1.

If the entropy distribution function $S(\psi)$ is taken so that $d\psi/dp$ is finite at p_0, or is infinite for $p = p_0$ to an order low enough so that the integral (13) still converges, the shock curve will end at a finite point P_0 in the (x, t)-plane. Under these conditions the shock decays after covering a finite distance in a finite time.

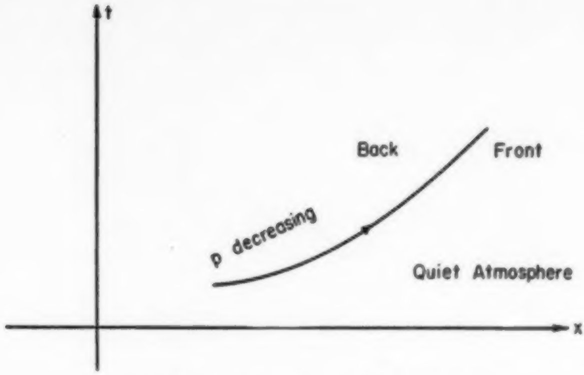


FIGURE 1. The shock curve for $d\psi/dp < 0$.

On the other hand, if $S(\psi)$ be chosen so that $d\psi/dp$ is infinite for $p = p_0$ and the integral (13) diverges, both x and t become infinite as p decreases to p_0. In this case the shock travels an infinite distance and requires an infinite time before it decays.

Decay after travelling an infinite distance in a finite time or a finite distance in an infinite time is accordingly not possible.

Let us write (5) in the form

$$S(\psi) - S_0 = c_p \log\left[\frac{(\gamma - 1)p + (\gamma + 1)p_0}{(\gamma + 1)p + (\gamma - 1)p_0}\left(\frac{p}{p_0}\right)^n\right], \quad S_0 = \text{const.}, \tag{15}$$

in which S_0 may be interpreted as the specific entropy of the quiet atmosphere in front of the shock. Expansion of the second member in powers of $p - p_0$ yields

$$S(\psi) - S_0 = A(p - p_0)^3 + \dots, \qquad A = n(1 - n^2)c_p/12p_0^3, \tag{16}$$

and differentiation gives

$$S'(\psi)\frac{d\psi}{dp} = c_p(\gamma - \gamma^{-1})\frac{(p - p_0)^2}{p[(\gamma - 1)p + (\gamma + 1)p_0][(\gamma + 1)p + (\gamma - 1)p_0]}, \tag{17}$$

where $S'(\psi)$ is the derivative of $S(\psi)$.

From (17) it is clear that (14) requires $S' < 0$, i.e., that the entropy distribution function be monotone decreasing. Furthermore the shock will require a finite time to decay if $S'(\psi)$, expressed as a function of p, vanishes to less than the third order in $p - p_0$; on the other hand, if $S'(\psi)$ vanishes to the third or a higher order in $p - p_0$, the time for decay is infinite.

4. Examples of shock decay. We illustrate the general principles in the previous section by simple examples.

For the first example, take

$$S(\psi) = S_0 - \psi,$$

so that, from (16)

$$\psi = -A(p - p_0)^3 + \dots, \quad \frac{d\psi}{dp} = -3A(p - p_0)^2 + \dots$$

from which and (13) it is clear that the shock decays in finite time.

For the second example, take

$$S(\psi) = S_0 + \psi^{-1},$$

for which

$$\psi = A^{-1}(p - p_0)^{-3} + \dots, \quad \frac{d\psi}{dp} = -3A^{-1}(p - p_0)^{-4} + \dots,$$

and it is clear again from (13) that shock decay now requires an infinite time.

More generally let us take

$$S(\psi) = S_0 - \frac{\psi^k}{k}, \qquad k = \pm 1, \pm 3, \dots.$$

Clearly $S'(\psi) < 0$ for all stated k, and its expansion in powers of $p - p_0$ is

$$S'(\psi) = -B(p - p_0)^{3(1-1/k)} + \dots,$$

where B is a positive constant. Thus $S'(\psi)$ will vanish to an order less than three at p_0 for $k = 1, 3, \dots$ and the shock decays in finite time; for $k = -1, -3, \dots S'(\psi)$ vanishes to a higher order than 3 at p_0 and decay requires an infinite time.

References

1. S. Chandrasekar, *On the decay of plane shock waves*, Aberdeen Proving Ground, BRL Report No. 423 (1943).
2. K. O. Friedrichs, *Formation and decay of shock waves*, Comm. on Appl. Math., *1* (1948), 211–245.
3. M. H. Martin, *The propagation of a plane shock into a quiet atmosphere*, Can. J. Math., *5* (1953), 37–39.

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland